COMPLEMENTATION IN THE LATTICE OF T_1-TOPOLOGIES

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Introduction. The purpose of this paper is to study the complementation problem in the lattice of T_1-topologies. In §1 it is shown that a large class of T_1-topologies do have complements. However, in general, the lattice of T_1-topologies is not a complemented lattice; a counterexample will be presented in §2.

Let Σ be the family of all topologies definable on an arbitrary set E. For $\tau_1 \in \Sigma$ and $\tau_2 \in \Sigma$, $\tau_1 < \tau_2$ if every set in τ_1 is in τ_2. Then τ_1 is said to be coarser than τ_2 and τ_2 finer than τ_1. Under this order, Σ is a complete lattice. The greatest element of Σ is the discrete topology, 1, and the least element is the trivial topology, 0. A topology with the property that the only finer topology is the discrete topology, is called an ultraspace on E.

The collection $\mathfrak{S}$ of subsets of E consisting of $\mathcal{P}(E-\{x\}) \cup \mathfrak{F}$, where $x \in E$, $\mathfrak{F}$ is a filter on E, and $\mathcal{P}(E-\{x\})$ is the power set of $E-\{x\}$, is a topology, denoted $\mathfrak{S}(x, \mathfrak{F})$. Fröhlich [2] proved that there is a one-to-one correspondence between ultraspaces on E and topologies of the form $\mathfrak{S}(x, \mathfrak{A})$; where $x \in E$ and $\mathfrak{A}$ is an ultrafilter on E, different from the principal ultrafilter at x, $\mathfrak{A}(x)$.

An ultraspace $\mathfrak{S}(x, \mathfrak{A})$ is a T_1-topology if and only if $\mathfrak{A}$ is a nonprincipal ultrafilter. In this case, $\mathfrak{A}$ contains no finite sets and $\mathfrak{S}(x, \mathfrak{A})$ is called a nonprincipal ultraspace. A topology on E is a T_1-topology if and only if it is the infimum of nonprincipal ultraspaces. Since any topology finer than a T_1-topology is a T_1-topology, the family Λ of T_1-topologies is a complete sublattice of the lattice of all topologies. The lattice Λ has a greatest element, 1, and a least element, the cofinite topology $\mathfrak{C}$, in which the empty set and complements of finite sets are open. Hartmanis [3] investigated the lattice of topologies and the lattice of T_1-topologies on a set E. He proved that Σ is complemented if E is finite. If E is finite, Λ consists of only one element and is trivially complemented. Hartmanis then asked if these lattices are also complemented if E is infinite. It has been shown that Σ is a complemented lattice even when E is infinite, Steiner [4].

1. **Complements for some T_1-topologies.** Topologies $\lambda_s = \mathfrak{C} \cup \{s\}$,

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where $\mathfrak{C}$ is the cofinite topology and $s \in E$, are called hyperplanes by Bagley [**1**]. He showed that the subset Λ_0 of Λ which consists of $\mathfrak{C}$ and all lattice joins of hyperplanes is a full set algebra on E and is maximal (in Λ) with respect to being uniquely complemented and containing $\alpha \vee \beta$ whenever it contains α and β.

Every hyperplane is the infimum of nonprincipal ultraspaces but no ultraspace is the supremum of hyperplanes.

THEOREM 1. *If τ is a T_1-topology on an infinite set $E = S \cup (E-S)$ such that $S \in \tau$, $\tau | S$ is discrete and the only open subsets of $E-S$ are in $\mathfrak{C}$, then τ has a lattice complement in Λ.*

PROOF. Let τ' be the union of sets of the form:

(i) $\{x\}$, for all $x \in E-S$,

(ii) U, for all $U \in \mathfrak{C}$.

Since τ' is finer than $\mathfrak{C}$, τ' is a T_1-topology. It is easily seen that $\tau \vee \tau' = 1$ since if $x \in S$ then $\{x\} \in \tau$ and if $x \in E-S$ then $\{x\} \in \tau'$.

Let $U \in \tau \wedge \tau'$, $U \neq \emptyset$. If $U \notin \mathfrak{C}$, then $U \in \tau'$ implies $U \subset E-S$. But $U \in \tau$ and $U \subset E-S$ imply $U \in \mathfrak{C}$. Thus $\tau \wedge \tau' = \mathfrak{C}$ and τ' is a complement for τ.

COROLLARY 1. *Every finite intersection of nonprincipal ultraspaces has a T_1-complement.*

PROOF. Let $\tau = \bigwedge_{i=1}^{N} \{\mathfrak{S}(x_i, \mathfrak{U}_i)\}$. Then $S = E - \{x_1, \cdots, x_N\}$ and $E - S \equiv \{x_1, \cdots, x_N\}$ satisfy the conditions of Theorem 1, that is, $\tau | S$ is discrete and $\emptyset$ is the only open subset of $E-S$.

COROLLARY 2. *Lattice joins of hyperplanes have T_1-complements.*

PROOF. Let $\tau = \bigvee_{s \in A} \lambda_s$. Then $\tau = \mathfrak{C} \cup \mathcal{P}(A)$, and $S = A$ and $E-S = E-A$ satisfy the conditions of Theorem 1.

The hyperplanes do not have unique complements in Λ. For example $\mathfrak{S}(s, \mathfrak{U})$ and $\mathfrak{S}(s, \mathfrak{V})$, $\mathfrak{U} \neq \mathfrak{V}$, are both T_1-complements for the hyperplane λ_s.

2. **Counterexample.** An example of a T_1-topology which has no complement in Λ will now be given.

Let τ be a T_1-topology on an infinite set $E = E_1 \cup E_2$, where E_1 and E_2 are infinite and disjoint, such that $E_1 \in \tau$, $E_2 \in \tau$, $\tau | E_1$ is cofinite and $\tau | E_2$ is discrete. Assume τ has a complement τ' in Λ.

For each $x \in E$, $\{x\} \in \tau \vee \tau'$. If $\{x\} \in \tau'$ for all $x \in E_1$, then $E_1 \in \tau \wedge \tau'$ but $E_1 \notin \mathfrak{C}$.

So assume there is some $x \in E_1$ such that $\{x\} \notin \tau'$. Then there is a $U \in \tau'$ such that $U \cap E_1$ is finite since $\{x\} = U \cap V$ for some $V \in \tau$ and

$E_1 - V$ is finite. But τ' is a T_1-topology so there is a $U^* \in \tau'$ such that $\emptyset \neq U^* \subset U$ and $U^* \cap E_1 = \emptyset$. Thus $U^* \cap E_2 = U^*$ and $U^* \in \tau \wedge \tau'$ but $U^* \notin \mathcal{C}$. Thus if $\tau \vee \tau' = 1$ then $\tau \wedge \tau' \neq \mathcal{C}$. Hence it has been verified that

THEOREM 2. *The lattice Λ of T_1-topologies on an infinite set E is not complemented.*

REFERENCES

1. R. W. Bagley, *On the characterization of the lattice of topologies*, J. London Math. Soc. **29-30** (1954–1955), 247–249.

2. O. Fröhlich, *Das Halbordnungssystem der topologischen Räume auf einer Menge*, Math. Ann. **156** (1964), 79–95.

3. J. Hartmanis, *On the lattice of topologies*, Canad. J. Math. **10** (1958), 547–553.

4. A. K. Steiner, *The lattice of topologies: structure and complementation*, Trans. Amer. Math. Soc. (to appear).

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SPACES WITH ACYCLIC POINT COMPLEMENTS

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1. **Introduction.** All homology groups will be singular homology with integer coefficients, reduced in dimension zero. If $0 \leqq n \leqq \infty$, a space X is *n-acyclic* if $H_q(X) = 0$ for all integers $q \leqq n$.

DEFINITION. A Hausdorff space M is an *A^n-space* if the complement of each point in M is n-acyclic.

The condition on a point x in M that $M - x$ be n-acyclic is similar to the notion that x be a non-r-cut point ($r \leqq n$), defined by R. L. Wilder [**9**, p. 218], using Čech theory.

Clearly spheres are A^∞-spaces. The object of this paper is to investigate to what extent A^n-spaces are like spheres. I wish to thank W. S. Massey for useful suggestions.

2. **Statement of results. Examples.** Open cells or closed cells of dimension $n+2$ are clearly A^n-spaces. Hilbert space l^2 is an A^∞-space; in fact by a theorem of Klee [**5**, p. 22], the complement of every compact subset of l^2 is homeomorphic to l^2 itself.

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