COMPLEMENTATION IN THE LATTICE OF $T_1$-TOPOLOGIES

A. K. STEINER

Introduction. The purpose of this paper is to study the complementation problem in the lattice of $T_1$-topologies. In §1 it is shown that a large class of $T_1$-topologies do have complements. However, in general, the lattice of $T_1$-topologies is not a complemented lattice; a counterexample will be presented in §2.

Let $\Sigma$ be the family of all topologies definable on an arbitrary set $E$. For $\tau_1 \in \Sigma$ and $\tau_2 \in \Sigma$, $\tau_1 < \tau_2$ if every set in $\tau_1$ is in $\tau_2$. Then $\tau_1$ is said to be coarser than $\tau_2$ and $\tau_2$ finer than $\tau_1$. Under this order, $\Sigma$ is a complete lattice. The greatest element of $\Sigma$ is the discrete topology, 1, and the least element is the trivial topology, 0. A topology with the property that the only finer topology is the discrete topology, is called an ultraspace on $E$.

The collection $\mathcal{G}$ of subsets of $E$ consisting of $\wp(E - \{x\}) \cup \mathcal{F}$, where $x \in E$, $\mathcal{F}$ is a filter on $E$, and $\wp(E - \{x\})$ is the power set of $E - \{x\}$, is a topology, denoted $\mathcal{G}(x, \mathcal{F})$. Fröhlich [2] proved that there is a one-to-one correspondence between ultraspaces on $E$ and topologies of the form $\mathcal{G}(x, \mathcal{A})$, where $x \in E$ and $\mathcal{A}$ is an ultrafilter on $E$, different from the principal ultrafilter at $x$, $\mathcal{A}(x)$.

An ultraspaces $\mathcal{G}(x, \mathcal{A})$ is a $T_1$-topology if and only if $\mathcal{A}$ is a nonprincipal ultrafilter. In this case, $\mathcal{A}$ contains no finite sets and $\mathcal{G}(x, \mathcal{A})$ is called a nonprincipal ultraspaces. A topology on $E$ is a $T_1$-topology if and only if it is the infimum of nonprincipal ultraspaces. Since any topology finer than a $T_1$-topology is a $T_1$-topology, the family $\Delta$ of $T_1$-topologies is a complete sublattice of the lattice of all topologies.

The lattice $\Delta$ has a greatest element, 1, and a least element, the cofinite topology $\mathcal{C}$, in which the empty set and complements of finite sets are open. Hartmanis [3] investigated the lattice of topologies and the lattice of $T_1$-topologies on a set $E$. He proved that $\Sigma$ is complemented if $E$ is finite. If $E$ is finite, $\Delta$ consists of only one element and is trivially complemented. Hartmanis then asked if these lattices are also complemented if $E$ is infinite. It has been shown that $\Sigma$ is a complemented lattice even when $E$ is infinite, Steiner [4].

1. Complements for some $T_1$-topologies. Topologies $\lambda_x = \mathcal{C} \cup \{x\}$,
where $C$ is the cofinite topology and $s \in E$, are called hyperplanes by Bagley [1]. He showed that the subset $\Lambda_0$ of $\Lambda$ which consists of $C$ and all lattice joins of hyperplanes is a full set algebra on $E$ and is maximal (in $\Lambda$) with respect to being uniquely complemented and containing $\alpha \lor \beta$ whenever it contains $\alpha$ and $\beta$.

Every hyperplane is the infimum of nonprincipal ultraspaces but no ultraspaces is the supremum of hyperplanes.

**Theorem 1.** If $\tau$ is a $T_1$-topology on an infinite set $E = S \cup (E - S)$ such that $S \subseteq \tau$, $\tau|S$ is discrete and the only open subsets of $E - S$ are in $C$, then $\tau$ has a lattice complement in $\Lambda$.

**Proof.** Let $\tau'$ be the union of sets of the form:

(i) $\{x\}$, for all $x \in E - S$,
(ii) $U$, for all $U \subseteq C$.

Since $\tau'$ is finer than $C$, $\tau'$ is a $T_1$-topology. It is easily seen that $\tau \lor \tau' = 1$ since if $x \in S$ then $\{x\} \in \tau$ and if $x \in E - S$ then $\{x\} \in \tau'$. Let $U \in \tau \land \tau'$, $U \neq \emptyset$. If $U \not\subseteq C$, then $U \in \tau'$ implies $U \subseteq E - S$. But $U \subseteq \tau$ and $U \subseteq E - S$ imply $U \subseteq C$. Thus $\tau \land \tau' = C$ and $\tau'$ is a complement for $\tau$.

**Corollary 1.** Every finite intersection of nonprincipal ultraspaces has a $T_1$-complement.

**Proof.** Let $\tau = \bigwedge_{i=1}^N \{\mathcal{S}(x_i, \mathcal{U}_i)\}$. Then $S = E - \{x_1, \ldots, x_N\}$ and $E - S = \{x_1, \ldots, x_N\}$ satisfy the conditions of Theorem 1, that is, $\tau|S$ is discrete and $\emptyset$ is the only open subset of $E - S$.

**Corollary 2.** Lattice joins of hyperplanes have $T_1$-complements.

**Proof.** Let $\tau = \bigvee_{s \in A} \lambda_s$. Then $\tau = C \cup \mathcal{P}(A)$, and $S = A$ and $E - S = E - A$ satisfy the conditions of Theorem 1.

The hyperplanes do not have unique complements in $\Lambda$. For example $\mathcal{S}(s, \mathcal{U})$ and $\mathcal{S}(s, \mathcal{V})$, $\mathcal{U} \neq \mathcal{V}$, are both $T_1$-complements for the hyperplane $\lambda_s$.

2. **Counterexample.** An example of a $T_1$-topology which has no complement in $\Lambda$ will now be given.

Let $\tau$ be a $T_1$-topology on an infinite set $E = E_1 \cup E_2$, where $E_1$ and $E_2$ are infinite and disjoint, such that $E_1 \subseteq \tau$, $E_2 \subseteq \tau$, $\tau|E_1$ is cofinite and $\tau|E_2$ is discrete. Assume $\tau$ has a complement $\tau'$ in $\Lambda$.

For each $x \in E$, $\{x\} \subseteq \tau \lor \tau'$. If $\{x\} \subseteq \tau'$ for all $x \in E_1$, then $E_1 \subseteq \tau \land \tau'$ but $E_1 \not\subseteq C$.

So assume there is some $x \in E_1$ such that $\{x\} \not\subseteq \tau'$. Then there is a $U \in \tau'$ such that $U \cap E_1$ is finite since $\{x\} = U \cap V$ for some $V \in \tau$ and
$E_1 - V$ is finite. But $\tau'$ is a $T_1$-topology so there is a $U* \in \tau'$ such that $\emptyset \neq U* \subseteq U$ and $U* \cap E_1 = \emptyset$. Thus $U* \cap E_1 = U*$ and $U* \in \tau \setminus \tau'$ but $U* \not\subseteq U$. Thus if $\tau \wedge \tau' = \emptyset$ then $\tau \vee \tau' \neq \emptyset$. Hence it has been verified that

**Theorem 2.** The lattice $\Lambda$ of $T_1$-topologies on an infinite set $E$ is not complemented.

**References**


University of New Mexico and
Texas Technological College

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**SPACES WITH ACYCLIC POINT COMPLEMENTS**

MICHAEL C. MCCORD

1. **Introduction.** All homology groups will be singular homology with integer coefficients, reduced in dimension zero. If $0 \leq n \leq \infty$, a space $X$ is $n$-acyclic if $H_q(X) = 0$ for all integers $q \leq n$.

**Definition.** A Hausdorff space $M$ is an $A^n$-space if the complement of each point in $M$ is $n$-acyclic.

The condition on a point $x$ in $M$ that $M - x$ be $n$-acyclic is similar to the notion that $x$ be a non-$r$-cut point ($r \leq n$), defined by R. L. Wilder [9, p. 218], using Čech theory.

Clearly spheres are $A^\infty$-spaces. The object of this paper is to investigate to what extent $A^n$-spaces are like spheres. I wish to thank W. S. Massey for useful suggestions.

2. **Statement of results. Examples.** Open cells or closed cells of dimension $n + 2$ are clearly $A^n$-spaces. Hilbert space $l^2$ is an $A^\infty$-space; in fact by a theorem of Klee [5, p. 22], the complement of every compact subset of $l^2$ is homeomorphic to $l^2$ itself.

Received by the editors October 26, 1965.