1. Let $F$ be a field of prime characteristic different from 2 or 3 and $L$ a Lie algebra over $F$ with an abelian Cartan subalgebra $H$. For $\alpha$ in $H^*$ (the dual space of $H$) set $L_\alpha = \{x \in L \mid [xh] = \alpha(h)x, \text{ for all } h \in H\}$, and as usual, if $L_\alpha \neq (0)$, $\alpha$ is called a root with respect to $H$ and $L_\alpha$ the root space for $\alpha$. We have $L_0 = H$ and $[L_\alpha L_\beta] \subseteq L_{\alpha + \beta}$. Seligman and Mills in [1] have called $L$ a Lie algebra of classical type if $L$ contains an abelian Cartan subalgebra $H$ and if $H$ and $L$ satisfy:

(i) $[LL] = L$.
(ii) $L$ has center $(0)$.
(iii) $L$ is a direct sum of subspaces $L_\alpha$.
(iv) If $\alpha$ is a nonzero root, then $[L_\alpha L_{-\alpha}]$ is one-dimensional.
(v) If $\alpha$ is a nonzero root and $\beta \in H^*$, then there is a positive integer $m$ such that $\beta + m\alpha$ is not a root.

Let $L$ be a Lie algebra over $F$ such that $L_K$ is of classical type, where $K$ is the algebraic closure of $F$. An extension field $P$ of $F$ is called a splitting field for $L$ provided $L_P$ is of classical type. We can now state the main theorem of this paper as:

**Theorem 1.1.** Every semisimple Lie algebra over $F$ with nondegenerate Killing form $(x, y) = \text{tr}(\text{ad}x)(\text{ad}y)$ has a separable splitting field.

Note that if $F$ is finite and $L$ has nondegenerate Killing form then every finite extension is separable, in particular one that splits $L$. Therefore, we may assume, in what follows, that $F$ is infinite.

2. **Lemma 2.1.** If $L$ is semisimple with nondegenerate Killing form, then $L$ contains a regular element $x$ such that the minimum polynomial of $\text{ad}(x)$ has the form:

$$\mu(\lambda) = \lambda \prod \alpha (\lambda - \alpha(x))$$

where the $\alpha(x)$ are distinct and different from zero in some extension $P$ of $F$.

**Proof.** Recall that an element $x$ in $L$ is regular provided the 0-space of $\text{ad}(x)$ has minimal dimension. If $x$ is regular in $L$ and $P$ is
an extension of $F$ then $x$ is regular in $L_P$. To see this, let $(u_1, u_2, \cdots, u_n)$ be a basis for $L$ and $(X_1, X_2, \cdots, X_n)$ be algebraically independent indeterminants. Let $F' = F(X_1, \cdots, X_n)$. Then $X = \sum X_i u_i$ is in $L_{P'}$ and the characteristic polynomial of $\text{ad}(X)$ is given by:

$$
\det(\lambda I - \text{ad}(X)) = \sum_{i=0}^{n} M_i(X)\lambda^i,
$$

where $M_i(X) \in F[X_1, \cdots, X_n]$. If $x = \sum_{i=1}^{n} \xi_i u_i$, then $M_i(\xi) = 0$ for $i < r$, where $r$ is the dimension of the zero space of $\text{ad}(x)$, and $M_r(\xi) \neq 0$. Also, since $x$ is regular, $M_i(\eta_1, \eta_2, \cdots, \eta_n) = 0$ for $i < r$ and all $\eta_i$ in $F$. Thus, since $F$ is infinite, $M_i(X)$ is zero for $0 \leq i < r$ and

$$
\det(\lambda I - \text{ad}(X)) = \sum_{r} M_i(X)\lambda^i.
$$

Now let $y = \sum \mu_j u_j$, $\mu_j$ in $P$, i.e. $y$ in $L_P$. Then $M_i(\mu) = 0$ for $0 \leq i < r$, so that the zero space of $\text{ad}(y)$ in $L_P$ has dimension greater than or equal to $r$. Therefore $x$ is regular in $L_P$ as claimed. As in the above, for generic element $X$ in $L_{P'}$, we have:

$$
\det(\lambda I - \text{ad}(X)) = \lambda(M_r(X) + M_{r+1}(X)\lambda + \cdots + M_n(X)\lambda^{n-r}),
$$

where $M_r(X) \neq 0$. Consider $g(X, \lambda)$, where:

$$
g(X, \lambda) = M_r(X) + M_r(X)\lambda + \cdots + M_n(X)\lambda^{n-r}.
$$

Then $g(X, \lambda) \in F[X_1, \cdots, X_n, \lambda] \subset F(X_1, \cdots, X_n)[\lambda] = F'[\lambda]$, and thus $g$ has a discriminant given by:

$$
D(X_1, \cdots, X_n) = \left( \prod_{i<j} (\rho_i - \rho_j) \right)^2,
$$

where $\rho_1, \rho_2, \cdots, \rho_{n-r}$ are all roots of $g(X, \lambda)$ as a polynomial in $\lambda$ in some splitting field over $F'$, multiple ones taken as many times as their multiplicity. Now, $D(X_1, \cdots, X_n)$ is a symmetric function of the roots and therefore is in the ring generated by the elementary functions of the roots, i.e. the ring generated by $M_r(X)$, $M_{r+1}(X)$, $\cdots$, $M_{n-r}(X)$. Thus, there exists a polynomial $Q(y_r, y_{r+1}, \cdots, y_n)$ with integral coefficients such that

$$
Q(M_r(X), \cdots, M_n(X)) = 0
$$

if and only if $g(X, \lambda)$ has repeated roots in its splitting field. Consider $g(X) = M_r(X) Q(M_r(X), \cdots, M_n(X))$. Then $g(X) \in F[X_1, \cdots, X_n]$ and if $g(X) \neq 0$ there exist elements $\xi_1, \cdots, \xi_n$ in $F$ such that $g(\xi_1, \cdots, \xi_n) \neq 0$. Suppose such an $n$-tuple exists and set $x = \sum \xi_i u_i$,
\[
\text{det}(\lambda I - \text{ad}(x)) = \lambda^r(M_r(\xi) + M_{r+1}(\xi)\lambda + \cdots + \lambda^{n-r}).
\]

\(M_r(\xi) \neq 0\) and \(M_r(\xi) + M_{r+1}(\xi)\lambda + \cdots + M_n(\xi)\lambda^{n-r}\) has distinct roots in a splitting field. Thus \(x\) is regular and the minimum polynomial of \(\text{ad}(x)\) has the form:

\[
\mu(\lambda) = \lambda \prod_a (\lambda - \alpha(x)),
\]

where \(\alpha(x)\) are distinct and different from zero. It remains to show that \(q(X) \neq 0\). For this, let \(H\) be a standard Cartan subalgebra in \(L_\xi\) and \(h_0 \in H\) be such that \(\alpha(h_0)\) are distinct and nonzero for all roots \(\alpha\) relative to \(H\). If \(h_0 = \sum \omega_i u_i, \omega_i \in K\), then \(g(\omega_1, \cdots, \omega_n)\) is not zero, so that \(g(X_1, \cdots, X_n) \neq 0\), as desired.

3. \textbf{Proof of Theorem 1.1.} Let \(L\) be a semisimple Lie algebra over \(F\) of dimension \(n\), with nondegenerate Killing form, and \(x\) a regular element, where the dimension of the zero-space of \(\text{ad}(x)\) is \(r\) and where the minimum polynomial of \(\text{ad}(x)\) has the form:

\[
\mu_{\alpha}(\lambda) = \lambda \prod_a (\lambda - \alpha(x)), \quad \alpha(x) \in F,
\]

with all \(\alpha(x)\) distinct, different from zero, and \(n-r\) in number. We will show that \(L\) is of classical type. Note that (ii) is satisfied by our hypotheses. Let \(H\) be the zero space of \(\text{ad}(x)\). Then \(H\) is the Cartan subalgebra of \(L_\xi\) which will play the role of satisfying the remaining axioms, and \(H\) has dimension \(r\). Since all \(\alpha(x)\) are distinct and characteristic roots of \(\text{ad}(h)\), the subspaces \(L_{\alpha(x)}\) corresponding to \(\alpha(x)\) have dimension one. Then we have

\[
L = H + \sum \alpha L_{\alpha}.
\]

Now for \(h \in H\), \([hx] = 0\) and if \(y \in L_{\alpha}\), \([yhx] = [[yx]h] = \alpha(x)[yh]\). Thus \([yh] \in L_{\alpha}\), i.e. \([L_{\alpha}H] \subseteq L_{\alpha}\) for \(\alpha(x) \neq 0\). Since \(L_{\alpha}\) is one dimensional this means that for \(e_{\alpha} \in L_{\alpha}\) and \(h \in H\), \([e_{\alpha}h] = \lambda(h)e_{\alpha}\). Set \(\alpha(h) = \lambda(h)\). Thus the characteristic roots of \(\text{ad}(h)\) are in the ground field and \(L_{\alpha}\) is a root space relative to \(H\). Furthermore, the restriction to \(H\) of the Killing form on \(L\) is nondegenerate. To see this, let \(h \in H\), \(e_{\alpha} \in L_{\alpha}\), for \(\alpha \neq 0\). Then \([e_{\alpha}h] = \alpha(h)e_{\alpha}\) and we can choose \(e_{\alpha}^{(1)} \in L_{\alpha}\) such that \([e_{\alpha}^{(1)}h] = e_{\alpha}\). Then \((h, e_{\alpha}) = (h, [e_{\alpha}^{(1)}h]) = ([hk], e_{\alpha}^{(1)}) = 0\). Thus \((H, L_{\alpha}) = 0\) for all \(\alpha \neq 0\) and the form must be nondegenerate on \(H\). It follows that \(H\) is abelian. In the case where \(F\) is algebraically closed
this is a result due to Zassenhaus. In the general case a field extension argument gives the result. This, together with (2) now shows that axiom (iii) holds.

Thus we have that $L$ contains an abelian Cartan subalgebra $H$, and that relative to a fixed basis for $L$, $\text{ad}(h)$ has a diagonal matrix for every $h \in H$. Next we note that if $\alpha$ is a root different from zero, then so is $-\alpha$. For, let $e_\alpha \in L_\alpha$, $e_\beta \in L_\beta$. Then, for some $h \in H$, $e_\alpha = [e_\alpha h]$, and $(e_\beta, e_\alpha) = (e_\beta [e_\beta e_\alpha], h) = 0$, unless $\beta = -\alpha$, since $[e_\beta e_\alpha] \in L_{\alpha + \beta}$. If $L_{-\alpha} = (0)$ we have $(L, e_\alpha) = 0$, a contradiction.

The nondegeneracy of the form $(x, y)$ on $H$ implies that for each $\alpha \in H^*$ there exists an $h_\alpha \in H$ such that $(h_\alpha, h) = \alpha(h)$, for all $h \in H$.

In what follows we shall have occasion to refer to results in Seligman’s Memoir [2] which we shall denote by $M$.

**Lemma 3.1 (M, Corollary 3.2).** If $e_\alpha \in L_\alpha$, $e_{-\alpha} \in L_{-\alpha}$, then $[e_{-\alpha} e_\alpha] = (e_{-\alpha}, e_\alpha) h$.

Since $L$ has nondegenerate form, every derivation of $L$ is inner [3]. Thus, for $x \in L$, $\text{ad}(x)^p$ is a derivation in $L$ and there exists a unique $y \in L$ such that $\text{ad}(x)^p = \text{ad}(y)$. Setting $x^p = y$, $L$ becomes a restricted Lie algebra over $F$ and the adjoint mapping is a restricted representation of $L$.

We turn now to a modification of two results due to Jacobson dealing with low-dimensional Lie algebras and their representations (M, Lemma 4.1 and 4.2). The modification involves replacing algebraic closure of the ground field with the fact that for the representation $U$ we have $U(h)$ is diagonalizable for all $h \in H$.

**Lemma 3.2.** Let $L$ be a two-dimensional Lie algebra over $F$ with basis elements $e$, $h$, and $[eh] = e$. Let $U$ be an irreducible representation of $L$ such that $U(h)$ and $U(e)^p$ are diagonalizable. Then either $U(e)^p = 0$ or $U$ is equivalent to the $p$-dimensional representation $W$:

$$W(e) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 \\ 0 & \cdots & 0 & 0 \end{pmatrix}, \quad W(h) = \begin{pmatrix} \lambda & 0 \\ \vdots & \ddots \\ 0 & \cdots & 0 & \lambda + p - 1 \\ 0 & \cdots & 0 & \lambda + 1 \end{pmatrix}.$$

**Proof.** Suppose $U(e)^p = U(e^p) \neq 0$. Then, since $U(e)^p$ and $U(h)^p - U(h)$ are diagonalizable by our assumptions, these matrices are scalar. For, if $V_1 = \{ v | v U(e)^p = \lambda v \}$, and $V_2 = \{ v | v (U(h)^p - U(h)) = \mu v \}$, then these are invariant subspaces of the representation space $V$ and since one of them is not zero for some $\lambda$ by diagonalizability one must be the whole space. Thus in each case, $U(e)^p = \sigma I$, $\sigma \in F$,
and \( U(h)^p - U(h) = \rho I, \rho \subseteq F \). Now, let \( \lambda \) be a characteristic root of \( U(h) \) and \( \nu \neq 0 \) such that \( \nu U(h) = \lambda \nu \). Then \( \nu U(e)^p = \sigma \nu \), and the space spanned by \( \{ \nu, \nu U(e), \cdots, \nu U(e)^{p-1} \} \) is an invariant subspace of dimension \( p \), thus the whole space \( V \). To see the invariance, we note:

\[
(3) \quad \nu U(e)^k U(h) = (\lambda + k)\nu U(e)^k, \quad 0 \leq k \leq p - 1.
\]

Now, relative to this basis for \( V \), the matrices of \( U(h) \) and \( U(e) \) have the form of the lemma.

**Lemma 3.3.** Let \( L \) be a three-dimensional Lie algebra over \( F \) with basis \( e, f, h \) and let \([ef] = h, [fh] = 0 = [eh]\). Let \( U \) be a nonzero irreducible representation of \( L \) such that \( U(e)^p = 0 \) and \( U(f)^p = 0 \) and \( U(h) \) is diagonalizable. Then \( \text{tr}(U(e) U(f)) = 0 \).

**Proof.** Since \( U(h) \) is diagonalizable and centralizes the representation we have \( U(h) = \lambda I \). If \( \lambda = 0 \), then \( U(e) U(f) = U(f) U(e) \). Since both \( U(e) \) and \( U(f) \) are nilpotent, so is \( U(e) U(f) \), and thus \( \text{tr}(U(e) U(f)) = 0 \). Suppose now that \( \lambda \neq 0 \) and let \( \nu \neq 0 \) be an element of the representation space \( V \) such that \( \nu U(f) = 0 \). Such a \( \nu \) exists since \( U(f) \) is nilpotent. Now, let \( K \) be the space spanned by \( \{ \nu, \nu U(e), \cdots, \nu U(e)^{p-1} \} \). Then \( K U(e) \subseteq K \) and \( KU(h) \subseteq K \). Furthermore, we have:

\[
(4) \quad \nu U(e)^k U(f) = \nu U(e)^{k-1} U(f) U(e) + \nu U(e)^{k-1} U(h).
\]

Actually, by induction we have:

\[
(5) \quad \nu U(e)^k U(f) = k \lambda (\nu U(e)^{k-1}), \quad k \geq 1.
\]

Thus, \( K = V \), the whole space, and thus \( \{ \nu, \nu U(e), \cdots, \nu U(e)^{p-1} \} \) is a basis for \( V \). The matrices relative to this basis are:

\[
U(e) = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 1 \\
0 & 0 & 0 & \cdots & 0
\end{bmatrix}, \quad U(f) = \begin{bmatrix}
0 & 0 & 0 & \cdots & 0 \\
\lambda & 0 & 0 & \cdots & 0 \\
0 & 2\lambda & \cdots & 0 \\
0 & 0 & \cdots & (p - 1)\lambda & 0
\end{bmatrix}
\]

and thus \( \text{tr}(U(e) U(f)) = 0 \) as claimed.

Using these lemmas together with our diagonalizability condition we can now prove the following analogues of the required theorems in \( M \).

**Theorem 3.1 (M, Theorem 4.1).** If \( \alpha \neq 0 \) is a root then \( e^\alpha_2 = 0 \).

**Proof.** For \( h \in H, [he^\alpha_2] = 0 \), thus \( e^\alpha_2 \subseteq H \). Choose \( h \) such that \( \alpha(h) = 1 \). Then \( \{ e_\alpha, h \} \) forms a two-dimensional Lie algebra \( L_1 \) as in Lemma 3.2. For the representation \( U(x) \) take \( \text{ad}_L(x) \). Then, the restriction of \( U \) to \( L_1 \) can be written in the form:
Furthermore, since \(\text{ad}_{L}(h)\) are diagonal for \(h \in H\), the same holds for \(\text{ad}_{L}(h)\) restricted to \(M\), \(M\) an irreducible \(L_{1}\) submodule of \(L\), and for the transformations induced by \(\text{ad}_{L}(h)\) in \(L/M\). Continuing this argument on \(L/M\) we see that \(U_{i}(h)\) is a diagonal matrix relative to a suitable basis for each \(h \in H\).

Now \((e_{a}^{\rho}, h) = \text{tr}(U(e_{a})^\rho U(h))\) and either \(U_{i}(e_{a})^\rho = 0\) or \(U_{i}(e_{a})\) has the form of Lemma 3.2. In any case, \(\text{tr}(U_{i}(e_{a})^\rho U_{i}(h))\) is zero so that \((e_{a}^{\rho}, h) = 0\). This holds whenever \(\alpha(h) \neq 0\). If \(\alpha(h) = 0\), let \(h_{1} \in H\) be chosen such that \(\alpha(h_{1}) \neq 0\). Then \(\alpha(h + h_{1}) \neq 0\) and \((e_{a}^{\rho}, h) = (e_{a}^{\rho}, h + h_{1}) - (e_{a}^{\rho}, h_{1}) = 0\). Thus, \((e_{a}^{\rho}, H) = 0\), which gives \(e_{a}^{\rho} = 0\).

**Theorem 3.2 (M, Theorem 4.2).** If \(\alpha \neq 0\) is a root, then \(\alpha(h_{a}) \neq 0\).

**Proof.** Suppose \(\alpha(h_{a}) = 0\) and \(e_{a} \neq 0\), \(e_{a} \in L_{a}, e_{-a} \neq 0, e_{-a} \in L_{-a}\) such that \((e_{a}, e_{-a}) = 1\). By Lemma 3.1 \([e_{-a}e_{a}] = h_{a}\). Let \(L_{1}\) be the algebra spanned by \(\{e_{a}, e_{-a}, h_{a}\}\). For the irreducible constituent \(U_{i}\) of the restriction of \(U = \text{ad}_{L}\) to \(L_{1}\) we have \(U_{i}(e_{a})^\rho = 0\) and \(U_{i}(e_{-a})^\rho = 0\). Thus by Lemma 3.3 we obtain \(\text{tr}(U_{i}(e_{-a})U_{i}(e_{a})) = 0\). Therefore, \(\text{tr}(U(e_{-a})U(e_{a})) = 0\), i.e. \((e_{-a}, e_{a}) = 0\), a contradiction.

Now, by Lemma 3.1 and Theorem 3.2 we have \([L_{a}L_{-a}]\) is one dimensional, giving us axiom (iv). Theorems 3.1 and 3.2 make it possible now to use the results of §5 of M. (For a complete proof of M, Lemma 5.1 see [1].) In particular, axiom (v) for our algebras is a consequence of Theorems 5.2 and 5.4 in M.

Finally, for axiom (i), we note that \([LL]_{K} = [L_{K}L_{K}]_{K} = L_{K}\), \(K\) the algebraic closure of \(F\). Hence we have:

\[\dim_{F}[LL] = \dim_{K}[LL]_{K} = \dim_{K}L_{K} = \dim_{F}L.\]

Therefore, \([LL] = L\). Thus, our algebra \(L\) is of classical type and this together with Lemma 2.1 proves Theorem 1.1.

**Bibliography**


**The Ohio State University**