A METRIZATION THEOREM

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A characterization of metrizable topological spaces in terms of subtopologies is given. First, several terms are defined in order to describe the pertinent subtopologies. Then, the characterization is readily established as a result of a metrization theorem due to Bing [1] and a metrization theorem due to Ceder [2].

DEFINITION 1. Let $(X, \mathfrak{J})$ be a topological space having the property that the intersection of an arbitrary number of open sets is open. Then $(X, \mathfrak{J})$ is called a *fundamental topological space.*

DEFINITION 2. Let $(X, \mathfrak{J})$ be a topological space with a base $\beta = \{B_j : j \in J\}$ such that for each $x \in \bigcap \{B_i : i \in I, I$ an arbitrary subset of $J\}$ there exists $B_x \in \beta$ where $x \in B_x \subset \bigcap \{B_i : i \in I\}$. Then, β is called a *fundamental base* for $\mathfrak{J}$.

REMARK. If $(X, \mathfrak{J})$ has a fundamental base for $\mathfrak{J}$, it is a fundamental space.

DEFINITION 3. Let $(X, \mathfrak{J})$ be a topological space. Let β_i, a subcollection of $\mathfrak{J}$, be a base for some topology, $\mathfrak{J}_i$, on X. Consider the subcollection of $\mathfrak{J}$, $\bar{\beta}_i$, where $\bar{\beta}_i = \{X - \overline{T}_i : T_i \in \mathfrak{J}_i$ and the closure is with respect to $\mathfrak{J}\}$. Then, $\bar{\beta}_i$ is a base for a topology on X, $\mathfrak{J}/\mathfrak{J}_i$. The topology, $\mathfrak{J}/\mathfrak{J}_i$ is called the *dual topology of* $\mathfrak{J}_i$ *relative to* $\mathfrak{J}$. $\bar{\beta}_i$ is called the *dual base of* β_i.

We state two metrization theorems, without proof, to be used in the establishment of a new metrization theorem. For other proofs see [1], [2].

A METRIZATION THEOREM OF BING. *A topological space is metrizable if and only if it is* T_1 *and regular and the topology has a* σ*-discrete base.*

A METRIZATION THEOREM OF CEDER. *A topological space* $(X, \mathfrak{J})$ *is metrizable if and only if it is* T_1 *and regular and the topology has a* σ*-closure preserving base,* $\beta = \bigcup_{i=1}^{\infty} \{B_i\}$, *such that for each* $x \in X$, *and* n, *a positive integer,* $\bigcap \{B : x \in B, B \in B_n\}$ *is a neighborhood of* x.

We now state a metrization theorem which characterizes a topological space, $(X, \mathfrak{J})$, as metrizable in terms of a countable family of bases, β_i, $i = 1, 2, \cdots$ for subtopologies, $\mathfrak{J}_i$ of $\mathfrak{J}$, and in terms of the dual bases, $\bar{\beta}_i$, $i = 1, 2, \cdots$. More precisely:

THEOREM. *A topological space* $(X, \mathfrak{J})$ *is metrizable if and only if it is* T_1 *and regular and has a base* $\beta = \bigcup_{i=1}^{\infty} \beta_i$ *where for each* i, β_i *is a fun-*

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damental base for a topology, $\mathfrak{I}_i$, *on* X, *and for each* i, *the dual base,* $\bar{\beta}_i$ *is a fundamental base for* $\mathfrak{I}/\mathfrak{I}_j$, *the dual topology of* $\mathfrak{I}_i$ *relative to* $\mathfrak{I}$.

PROOF. Let $(X, \mathfrak{I})$ be a topological space satisfying the hypotheses above. Then, we shall show that $(X, \mathfrak{I})$ satisfied the hypotheses of Ceder's Theorem. Let $\beta = \bigcup_{i=1}^{\infty} \beta_i$ be the given base for $\mathfrak{I}$. Then, for each $x \in X$, n a positive integer, $\bigcap \{B : x \in B,\ B \in \beta_n\}$ is a neighborhood of x in $\mathfrak{I}_n \subset \mathfrak{I}$. Also, $X - \bigcup \{\overline{B}_i : B_i \in \beta_n' \subset \beta_n\} = \bigcap \{X - \overline{B}_i\} \in \mathfrak{I}/\mathfrak{I}_i \subset \mathfrak{I}$. Therefore, $\bigcup \{\overline{B}_i\}$ is closed. Hence, $\bigcup \{\overline{B}_i\} = \mathrm{cl}[\bigcup B_i]$. Thus, β_n is a closure preserving family.

Now, let $(X, \mathfrak{I})$ satisfy the hypotheses of Bing's Theorem, and let $\beta = \bigcup_{i=1}^{\infty} \beta_i$ be the σ-discrete base. We define $\beta_i' = \beta_i \cup \{X\}$, $i = 1, 2, \cdots$. It is apparent that β_i' is a fundamental base for some subtopology, $\mathfrak{I}_i'$, of $\mathfrak{I}$ for each i. It remains only to show that $\bar{\beta}_i'$ is a fundamental base, also. Now, $\bigcap_\alpha \{X - \overline{T}_\alpha : T_\alpha \in \mathfrak{I}_i'\} = X - \bigcup_\alpha \overline{T}_\alpha = X - \mathrm{cl}[\bigcup_\alpha T_\alpha] \in \mathfrak{I}/\mathfrak{I}_i'$, since β_i' is a closure preserving subfamily of $\mathfrak{I}$ implies that $\mathfrak{I}_i'$ is also a closure preserving subfamily of $\mathfrak{I}$. Thus, $\bar{\beta}_i'$ is a fundamental base.

REFERENCES

1. R. H. Bing, *Metrization of topological spaces*, Canad. J. Math. **3** (1951), 175–186.

2. J. G. Ceder, *Some generalizations of metric spaces*, Pacific J. Math. **11** (1961), 105–125.

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