argument (see [1, Lemma 2.4]) that this last series is finite. As a corollary to the theorem we note that

$$\sum_{n=1}^{\infty} (-1)^{n-1}P(|S_n| > n\epsilon)$$

is convergent for all \(\epsilon > 0\). In [2] it is shown that this series is absolutely convergent for all \(\epsilon > 0\) if and only if \(EX_1 = 0\).

References


University of Missouri and
University of New Mexico

---

**THE BOUNDARY OF THE RANGE OF A VECTOR MEASURE**

**VOLKER BAUMANN**

Let \((X, \mathcal{S})\) be a measure space, \(\mu_i (i = 1, \ldots, n)\) signed measures on \((X, \mathcal{S})\). Then \(\mu = (\mu_1, \ldots, \mu_n)\) is a \((n\)-dimensional) vector measure on \((X, \mathcal{S})\), \(\mu\) is finite and purely nonatomic if every \(\mu_i\) is finite and purely nonatomic, respectively. Consider the range of a finite \(n\)-dimensional vector measure as a subset of the \(n\)-dimensional Euclidean space \(E^n\). A. Liapounoff [4] and P. R. Halmos [2] have shown:

1. The range of a finite vector measure is closed,
2. the range of a finite and purely nonatomic vector measure is convex.

For any (infinite) vector measure \(\mu\) call \(R = \{\mu(M) : M \in \mathcal{S} \text{ and } \mu(M) \text{ finite}\}\) the finite range of \(\mu\). Then it is an immediate consequence of (2) that

3. the finite range of a purely nonatomic vector measure is convex.

Two simple examples due to R. Borges [1] however show that there are purely nonatomic as well as purely atomic vector measures the finite range of which is not closed:

(a) \(\mathcal{S}\) is the \(\sigma\)-ring of the one-dimensional Lebesgue sets, \(\mu_1\) the Lebesgue measure and \(\mu_2(M) = \int_M \exp(-z^2)dz, M \in \mathcal{S}\). The positive

Received by the editors January 2, 1964.
x_1\)-axis does not belong to the finite range \( R \) of \((\mu_1, \mu_2)\), but it belongs to the closure \( \overline{R} \) of \( R \).

(b) \( S \) is the \( \sigma \)-ring of all subsets of the set of positive integers, \( \mu \) the one-dimensional vector measure defined by \( \mu(M) = \sum_{n \in M} (2 - 2^{-n}) \), \( M \in S \). The set of the even positive integers does not belong to \( R \), but it belongs to \( \overline{R} \).

As to boundary points of the finite range \( R \) of a vector measure lying moreover on the boundary of the convex hull of \( R \) here it will be proved the

**THEOREM.** Let \( \pi \) be a supporting hyperplane of the finite range \( R \) of a vector measure, \( \overline{R} \) the closure of \( R \). If \( \pi \cap \overline{R} \) is bounded, then \( \pi \cap \overline{R} \subset R \).

Restricted to purely nonatomic vector measures the theorem was proved by R. Borges [1].

**Proof.** Given \( \pi \), there is a real linear function \( L \) on \( E^n \) and a real number \( u \) such that \( \pi = \{ \xi \in E^n : L(\xi) = u \} \) and

\[
(4) \quad u = \inf \{ \nu(M) : M \in S_0 \},
\]

where \( \nu(M) = L(\mu(M)) \) and \( S_0 = \{ M \in S : \mu(M) \text{ finite} \} \). Let \( \xi \in \pi \cap \overline{R} \). Then there is a sequence \( \{ M_j \} \), \( M_j \in S_0 \), \( \lim \mu(M_j) = \xi \). We can assume that

\[
(5) \quad \nu(M_j) \leq u + 2^{-j} \quad (j = 1, 2, \ldots).
\]

Write \( P_j^k = \bigcup_{i=j}^{j+k} M_i \). Since \( S_0 \) is a ring (see [3, p. 19 and 119]) and \( \nu \) is additive on \( S_0 \) it is easy to verify by induction for \( k \), using (4) and (5), that

\[
(6) \quad u \leq \nu(P_j^k) \leq u + 2^{-j} \sum_{m=0}^{k} 2^{-m} \quad (j = 1, 2, \ldots ; k = 0, 1, \ldots).
\]

Therefore for all \( j, k \)

\[
(6) \quad u \leq \nu(P_j^k) \leq u + 2^{1-j}.
\]

Assume that \( P = \bigcup_{j=1} M_j \in S_0 \). Then, given \( c > 0 \), for every \( j \) there is a nonnegative integer \( k \) such that

\[
(7) \quad ||\mu(P_j^k)|| \geq c,
\]

where \( || \cdot || \) denotes the Euclidean norm (see [3, p. 120]). Since for every \( i \) the upper or the lower variation of \( \mu_i \) is bounded, say by \( w_i/2 \), the total variation \( ||\mu_i|| \) of \( \mu_i \) satisfies for every \( M \in S_0 \) the inequality \( ||\mu_i||(M) \leq ||\mu_i(M)|| + w_i \), especially \( ||\mu_i(P_j^{k+1} - P_j^k)|| \leq ||\mu_i||(M_{j+k+1}) \leq ||\mu_i(M_{j+k+1})|| + w_i \) and therefore for every \( j, k \)
\[ \|\mu(F_{j}^{k+1} - F_{j}^{k})\| \leq w, \]

where \( w = n \sup \{ |\mu_{i}(M_{j})| + w_{i}; i = 1, \ldots, n; j = 1, 2, \cdots \} \) is finite and independent of \( j \) and \( k \). Also \( \|\mu(F_{j}^{0})\| = \|\mu(M_{j})\| \leq w \). For every \( j \) let \( k(j) \) be the smallest nonnegative integer \( k \) satisfying (7). Then

\[ c \leq \|\mu(F_{j}^{k(j)})\| \leq c + w \quad (j = 1, 2, \cdots). \]

A closed and bounded subset of the \( E^{n} \) is compact. Thus it follows from (6) and (8) that there is an \( \eta \in \pi \cap \overline{R}, \|\eta\| \geq c; c \) was given arbitrarily positive, in contradiction to the boundedness of \( \pi \cap \overline{R} \). Hence \( F \in S_{0} \).

The restriction \( \mu' \) of \( \mu \) to the \( \sigma \)-ring \( S' = \{ F \cap M: M \in S \} \) is a finite vector measure. Since \( M_{j} \in S', j = 1, 2, \cdots, \xi \) is a cluster point of the range of \( \mu' \). According to (1), \( \xi \in \overline{R} \).

References


Universität Köln, Germany