ON A CLASS OF MEROMORPHIC FUNCTIONS

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In his paper [1] F. Gross considers functions \( f(z) \) and \( g(z) \) meromorphic in the plane and satisfying

\[ f^n + g^n = 1, \]

where \( n \) is a fixed integer. For \( n=2 \) he shows that all meromorphic solutions of (1) are of the form

\[ f = \frac{2\beta}{1 + \beta^2}, \quad g = \frac{1 - \beta^2}{1 + \beta^2}, \]

where \( \beta \) is meromorphic. In this case one may even obtain entire solutions, e.g. \( f = \sin z \), \( g = \cos z \), \( \beta = \tan (z/2) \). Gross also shows that for \( n>2 \) there are no entire solutions of (1), while for \( n>3 \) there are no meromorphic solutions.

Now the equation \( w^3 + z^3 = 1 \) defines an algebraic function whose Riemann surface has genus 1, and there is accordingly a uniformization by elliptic functions. If \( \wp(z) \) is the Weierstrass elliptic function with periods \( \omega_1, \omega_2 \) satisfying

\[ (\wp')^2 = 4\wp^3 - g_2\wp - g_3, \quad g_2, g_3 \text{ constants}, \]

then (cf. [2, p. 227]) \( \omega_1 \) and \( \omega_2 \) may be chosen so that

\[ g_2 = 0, \quad g_3 = 1. \]

With this \( \wp(z) \) we find that

\[ f(z) = \frac{1}{2} + \frac{\wp'(z)}{(12)^{1/2}} \bigg/ \wp(z), \]

\[ g(z) = \frac{1}{2} - \frac{\wp'(z)}{(12)^{1/2}} \bigg/ \wp(z), \]

satisfy

\[ f^3 + g^3 = 1. \]

The formulas (2) differ from the analogous formulae in [1], which seem to contain an error.

With the aid of the functions in (2) one may verify a conjecture made by F. Gross in [1], viz. that meromorphic solutions of (3) are

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necessarily elliptic functions of entire functions. We shall prove

**Theorem 1.** Any functions $F(z), G(z)$, which are meromorphic in the plane and satisfy

$$F^3 + G^3 = 1,$$

have the form

$$F = f(h(z)), \quad G = \eta g(h(z)) = \eta f(-h(z)) = f(-\eta^2 h(z)),$$

where $f$ and $g$ are the elliptic functions in (2), $h(z)$ is an entire function of $z$ and $\eta$ is a cube-root of unity.

**Proof.** Write $\rho = \exp(2\pi i/3)$. If $F$ and $G$ are meromorphic solutions of (4), then since $F = (1 - G^3)^{1/3}$ is single-valued, it follows that the multiplicity of any solution $z$ of $G(z) = \rho, \rho^2$ or 1, is a multiple of 3.

We shall need to discuss the singularities of the inverse function $f^{-1}(w)$ of the $f(z)$ in (2). Since $f(z)$ is a doubly periodic function it has neither finite nor infinite asymptotic values and hence, by Iversen’s theorem, all the singularities of $f^{-1}(w)$ are algebraic. We prove that these singularities all lie over $w = \rho, \rho^2$ or 1. First note that $\wp(z)$ has double poles at the points $m\omega_1 + n\omega_2, m, n$ integral, and so $f(z)$ has single poles at these points. $\wp(z)$ has poles nowhere else, so that the other poles of $f(z)$ are at the zeros of $\wp(z)$ and hence there are two simple ones in each period parallelogram, since $\wp(z)$ takes each value twice in such a parallelogram, while $\wp(z) = 0$ implies

$$(\wp')^2 = 4\wp^3 - 1 = -1 \neq 0.$$ 

Thus altogether $f(z)$ has three simple poles in each period parallelogram $S$, while by differentiation $f'(z)$ has three double poles and thus $f$ and $f'$ take each value three or six times respectively, in $S$. Now $f^3 + g^3 = 1$, so that by the first remarks of this proof $f = \rho, \rho^2$ and 1 at least triply at each solution. Thus $f$ takes each value $\rho, \rho^2$ and 1 precisely at one point in $S$, the derivative $f'$ having a double zero at each of these points and at no other points of $S$. Thus all singularities of $f^{-1}(w)$ lie over $w = \rho, \rho^2$ and 1. In particular $w = \infty$ is a regular point of each branch of $f^{-1}(w)$.

We return to the consideration of $F, G$ satisfying (4), and in the neighborhood of any value $z_0$, such that $w = F(z_0) \neq \rho, \rho^2, 1$, we take any branch of $f^{-1}(w)$ and form the regular function element

$$h(z) = f^{-1}(F(z)).$$

Now $h(z)$ may be continued analytically along any curve $\gamma$ in the plane without restriction. Obviously the continuation can only fail
when \( \gamma \) reaches a point \( z_1 \) such that \( w_1 = F(z_1) = \rho, \rho^2 \) or 1. Denote by \( \gamma_1 \) the arc of \( \gamma \) between \( z_0 \) and \( z_1 \), exclusive of the end point \( z_1 \). Then \( h(z) \) is regular along \( \gamma_1 \) and for each point \( z \) on \( \gamma_1 \) there is a branch of \( f_{-1} \) such that \( f_{-1}(F(z)) = h(z) \). Now \( z_1 \) is a \( 3k \)-fold solution of \( F(z_1) = w_1 \), so \( F(z) = w_1 + \{ \phi(z) \}^3 \), where \( \phi(z) \) is a regular function in the neighborhood \( N: |z - z_1| < \delta, \delta > 0 \), and satisfies \( \phi(z_1) = 0 \). We may suppose \( \delta \) chosen so small that for \( z \) in \( N \) we have \( |F(z) - w_1| < 1 \).

Now in the neighborhood \( M: |w - w_1| < 1 \), the only branch points of \( f_{-1}(w) \) lie over \( w = w_1 \). For some \( z \) in \( \gamma_1 \cap N \) we form \( w = F(z) \) in \( M \) and choose the branch \( f_{-1}(w) \) such that \( f_{-1}(F(z)) = h(z) \). We note that for neighboring values \( z \) we obtain the same branch \( f_{-1}(w) \), which indeed has an expansion

\[
f_{-1}(w) = \lambda + P((w - w_1)^{1/3}), \quad |w - w_1| < (3)^{1/2},
\]

where \( \lambda \) is a constant and \( P(t) \) is a convergent power series in \( t \). Thus we must have for all \( z \) in \( \gamma_1 \cap N \), using \( F = w_1 + \phi^3 \), an expression

\[
h(z) = \lambda + P(\mu \phi),
\]

where \( \mu \) is a fixed 3rd root of unity and \( \phi \) is regular in \( N \). This expression gives a regular continuation of \( h(z) \) over the value \( z_1 \). Thus we have verified that \( h(z) \) can be continued throughout the plane to give (by the monodromy theorem) a function regular in the plane i.e. an entire function.

We now have \( F(z) = f(h(z)) \) and

\[
F^3 + G^3 = 1, \quad f^3 + g^3 = 1, \quad f(h)^3 + g(h)^3 = 1 = F^3 + g(h)^3.
\]

Hence \( G^3 = g(h)^3, \quad G = \eta g(h) \), where \( \eta \) is (since \( G, g(h) \) are regular) a fixed third root of unity. Since \( \phi \) is even and \( \phi' \) is odd we have \( f(-z) = g(z) \) and \( G \) can also be written \( \eta f(-h) \).

We remark finally that (cf. [2, p. 168]) \( \phi(z) \) has an expansion

\[
\phi(z) = \frac{1}{z^2} + \sum_{n=3}^{\infty} C_n z^{2n-2},
\]

where

\[
(n - 3)(2n + 1)C_n = 3(C_2C_{n-2} + C_3C_{n-3} + \cdots + C_{n-2}C_2),
\]

\[
n = 4, 5, 6, \ldots,
\]

\[
C_2 = \frac{1}{20} g_2, \quad C_3 = \frac{1}{28} g_3.
\]
Since \( g_2 = 0, g_3 = 1 \), it is easy to prove inductively that \( C_n = 0 \) unless \( n \equiv 0 \) modulo 3. Substitution in (2) shows that \(zf(z)\) is a function of \( z^3\), so that

\[
f(\eta z) = \eta^2 f(z), \quad \eta^3 = 1.
\]

This shows that \( f(-\eta^2 h(z)) = \eta f(-h(z)) \), and the proof of the equivalence of the various expressions for \( G \) in (5) is complete.

**References**


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**ON THE BOUNDARY BEHAVIOR OF FUNCTIONS MEROMORPHIC IN THE UNIT DISK**

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1. **Introduction.** Let \( f(z) \) be meromorphic in \( D: \{ |z| < 1 \} \), and suppose that the values assumed by \( f(z) \) in \( D \) lie in a domain \( G \) whose boundary \( \Gamma \) has positive logarithmic capacity. Then \( f(z) \) is of bounded characteristic in \( D \) and has finite radial limits \( f(e^{i\theta}) = \lim_{r \to 1} f(re^{i\theta}) \) at almost all points \( e^{i\theta} \) on \( C: \{ |z| = 1 \} \). (For this and more general theory of meromorphic functions, see [4, pp. 208 ff.].) The class of functions satisfying these conditions and having the additional property that \( f(e^{i\theta}) \) belongs to \( \Gamma \) almost everywhere on \( C \) has been studied by O. Lehto [3] and D. A. Storvick [6], who called it class (L).

If \( A \) is a sequence of points in \( D \) satisfying \( \sum_{a \in A} (1 - |a|) < \infty \), the Blaschke product with respect to \( A \) in \( D \) is the function \( B(z; A) = \prod_{a \in A} [(1 - |a|) / a(a - \bar{a}z)] \). The present note arises from a suggestion by Professor Storvick that the following theorem, established in [1], be extended to functions in class (L). Here we denote by \( A' \) the derived set of \( A \).

**Theorem 1.** Let \( E \) be a set on \( C. A \) necessary and sufficient condition

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