ON CUBIC TRANSFORMATIONS OF ORTHOGONAL POLYNOMIALS

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After investigating generalizations of the quadratic transformation relating Hermite and Laguerre polynomials

$$H_{2n}(x) = (-1)^n 2^{2n} n!\, L_n^{(-1/2)}(x^2),$$

T. S. Chihara asked [1] for an example of a pair of orthogonal polynomial sets $\{R_n(x)\}$ and $\{S_n(x)\}$ such that $R_{3n}(x) = S_n(z)$ where z is a cubic in x. It is the purpose of this paper to exhibit such examples.

Let $\{S_n(z)\}$ be a set of monic polynomials with $S_n(-z) = (-1)^n S_n(z)$. For this set to be a set of orthogonal polynomials, it is both necessary and sufficient (Shohat [**3**, pp. 454 and 456]) that there exist a sequence of positive numbers $\{B_n\}$ such that

$$zS_n(z) = S_{n+1}(z) + B_n S_{n-1}(z), \qquad n = 1, 2, \cdots, \tag{1}$$

or equivalently,

$$(x^3 + \beta x)S_n(x^3 + \beta x) = S_{n+1}(x^3 + \beta x) + B_n S_{n-1}(x^3 + \beta x), \qquad n = 1, 2, \cdots. \tag{2}$$

We look for a monic set of orthogonal polynomials $\{R_n(x)\}$ with $R_n(-x) = (-1)^n R_n(x)$ and with a subset $\{R_{3n}(x)\}$ that satisfies a recurrence relation with the same coefficients,

$$(x^3 + \beta x)R_{3n}(x) = R_{3n+3}(x) + B_n R_{3n-3}(x), \qquad n = 1, 2, \cdots. \tag{3}$$

Such a set of orthogonal polynomials exists if (3) can be formed by iterating

$$xR_n(x) = R_{n+1}(x) + b_n R_{n-1}(x), \qquad n = 1, 2, \cdots, \tag{4}$$

the condition (with $b_n > 0$, $n = 1, 2, \cdots$) that is equivalent to the condition that $\{R_n(x)\}$ is an orthogonal set.

Iterating (4), we may write successively

$$xR_{3n}(x) = R_{3n+1}(x) + b_{3n}R_{3n-1}(x),$$

$$x^2R_{3n}(x) = (R_{3n+2}(x) + b_{3n+1}R_{3n}(x)) + b_{3n}(R_{3n}(x) + b_{3n-1}R_{3n-2}(x)),$$

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$$\begin{aligned} x^3R_{3n}(x) = {} & (R_{3n+3}(x) + b_{3n+2}R_{3n+1}(x)) \\ & + (b_{3n+1} + b_{3n})(R_{3n+1}(x) + b_{3n}R_{3n-1}(x)) \\ & + b_{3n}b_{3n-1}(R_{3n-1}(x) + b_{3n-2}R_{3n-3}(x)) \end{aligned}$$

and thus

$$\begin{aligned} (x^3 + \beta x)R_{3n}(x) = {} & R_{3n+3}(x) \\ & + (b_{3n+2} + b_{3n+1} + b_{3n} + \beta)R_{3n+1}(x) \\ & + (b_{3n+1} + b_{3n} + b_{3n-1} + \beta)b_{3n}R_{3n-1}(x) \\ & + b_{3n}b_{3n-1}b_{3n-2}R_{3n-3}(x), \qquad n = 1, 2, \cdots \end{aligned}$$

is the same recurrence relation as (3) provided β and the b_n are such that

$$\begin{aligned} b_{3n+2} + b_{3n+1} + b_{3n} + \beta &= 0, \qquad n = 1, 2, \cdots, \\ b_{3n+1} + b_{3n} + b_{3n-1} + \beta &= 0, \qquad n = 1, 2, \cdots, \\ b_{3n}b_{3n-1}b_{3n-2} &= B_n, \qquad n = 1, 2, \cdots. \end{aligned} \tag{5}$$

Thus, assuming these three conditions (5) on the recurrence relations (1) and (4), the polynomial sets $\{S_n(x^3+\beta x)\}$ and $\{R_{3n}(x)\}$ satisfy the same recurrence relations ((2) or (3)) and if we further place the initial restrictions

$$S_0(x^3 + \beta x) = R_0(x) \tag{6}$$

and

$$S_1(x^3 + \beta x) = R_3(x), \tag{7}$$

then we will have

$$S_n(x^3 + \beta x) = R_{3n}(x), \qquad n = 0, 1, 2, \cdots.$$

Condition (6) is satisfied because our assumption that polynomial sets are monic implies $S_0(z) = R_0(x) = 1$. Since $S_1(z) = z$, condition (7) becomes $x^3+\beta x = R_3(x)$ or, because $R_3(x) = xR_2(x) - b_2R_1(x) = x^3 - (b_1+b_2)x$, we may replace (7) by the condition

$$b_1 + b_2 + \beta = 0. \tag{8}$$

Thus, after combining (8) and the three conditions (5), we have established the

THEOREM. *If $\{S_n(z)\}$ is a set of orthogonal polynomials with $S_0(z) = 1$, $S_1(z) = z$ and*

$$zS_n(z) = S_{n+1}(z) + B_nS_{n-1}(z), \qquad n = 1, 2, \cdots \tag{1}$$

and if there exist numbers β and b and a sequence of positive numbers $\{b_n\}$ such that the four conditions

$$\begin{aligned} b_1 + b + \beta &= 0, \\ b_{3n-1} &= b, & n &= 1, 2, \cdots, \\ b_{3n+1} + b_{3n} + b + \beta &= 0, & n &= 1, 2, \cdots, \\ b_{3n} b b_{3n-2} &= B_n, & n &= 1, 2, \cdots, \end{aligned}$$

are met, then the polynomials $\{R_n(x)\}$ defined by $R_0(x)=1$, $R_1(x)=x$, and

$$xR_n(x) = R_{n+1}(x) + b_n R_{n-1}(x), \qquad n = 1, 2, \cdots \tag{4}$$

form an orthogonal set with the property

$$S_n(x^3 + \beta x) = R_{3n}(x), \qquad n = 0, 1, \cdots. \tag{9}$$

After this theorem, one naturally inquires about expressing the polynomials of the sets $\{R_{3n+1}(x)\}$ and $\{R_{3n+2}(x)\}$ in terms of the polynomials of the set $\{S_n(x^3+\beta x)\}$. We can express $R_{3n+1}(x)$ in terms of $R_{3n}(x)$ and $R_{3n+3}(x)$ by successively combining recurrence relations:

$$\begin{aligned} xR_{3n+1}(x) &= R_{3n+2}(x) + b_{3n+1}R_{3n}(x), \\ x^2R_{3n+1}(x) &= (R_{3n+3}(x) + b_{3n+2}R_{3n+1}(x)) + xb_{3n+1}R_{3n}(x), \\ (x^2 - b_{3n+2})R_{3n+1}(x) &= R_{3n+3}(x) + xb_{3n+1}R_{3n}(x). \end{aligned}$$

Thus we are led to the

COROLLARY. *For polynomial sets meeting the hypotheses of the theorem,*

$$R_{3n+1}(x) = (x^2 - b)^{-1}(S_{n+1}(x^3 + \beta x) + xb_{3n+1}S_n(x^3 + \beta x)), \qquad n = 0, 1, \cdots,$$

$$R_{3n+2}(x) = (x^2 - b)^{-1}(xS_{n+1}(x^3 + \beta x) + b_{3n+2}b_{3n+1}S_n(x^3 + \beta x)), \qquad n = 0, 1, \cdots.$$

A particularly simple set of constants meeting the conditions of the theorem is $b_n=B_n=1$ for $n>1$, $\beta=-3$, and $b_1=B_1=2$. For this choice of constants, the two sets $\{R_n(x)\}$ and $\{S_n(x)\}$ are identical and $R_{3n}(x)=R_n(x^3-3x)$, $n=0, 1, \cdots$. The recurrence relation

$$\begin{aligned} xR_n(x) &= R_{n+1}(x) + R_{n-1}(x), \qquad n = 1, 2, \cdots, \\ xR_1(x) &= R_1(x) + 2R_0(x) \end{aligned}$$

and the polynomials $\{R_n(x)\}$ when transformed by $R_0(2x)=T_0(x)$ and $R_n(2x)=2T_n(x)$, $n=1, 2, \cdots$, lead us to the nonmonic set $\{T_n(x)\}$ defined by

$$T_0(x) = 1, \qquad T_1(x) = x,$$
$$2xT_n(x) = T_{n+1}(x) + T_{n-1}(x), \qquad n = 1, 2, \cdots.$$

But this is a definition of the Tchebichef polynomials of the first kind [2, (10.11)]. Thus we have shown that for the Tchebichef polynomials of the first kind $\{T_n(x)\}$,

$$\begin{aligned} T_{3n}(x) &= T_n(4x^3 - 3x), && n = 0, 1, \cdots, \\ T_{3n+1}(x) &= (4x^2 - 1)^{-1}(T_{n+1}(4x^3 - 3x) + 2xT_n(4x^3 + 3x)), \\ &&& n = 0, 1, \cdots, \\ T_{3n+2}(x) &= (4x^2 - 1)^{-1}(2xT_{n+1}(4x^3 - 3x) + T_n(4x^3 - 3x)), \\ &&& n = 0, 1, \cdots. \end{aligned} \tag{10}$$

The cubic transformation (10), which may be written as $T_{3n}(x) = T_n(T_3(x))$, is a special case of the transformation of arbitrary degree

$$T_{mn}(x) = T_m(T_n(x)), \qquad m, n = 0, 1, \cdots,$$

that is an easy consequence of the definition $T_n(\cos\theta)=\cos n\theta$.

For a less trivial example of the theorem, let the $S_n(z)$ be the monic Legendre polynomials defined in terms of the ordinary Legendre polynomials $P_n(z)$ by $2^n(1/2)_n(n!)^{-1}S_n(z)=P_n(z)$, $n=0, 1, \cdots$, or equivalently, defined directly by

$$S_0(z) = 1, \quad S_1(z) = z,$$
$$xS_n(z) = S_{n+1}(z) + \frac{n^2}{(2n+1)(2n-1)} S_{n-1}(z), \qquad n = 1, 2, \cdots.$$

Here we have

$$B_n = \frac{n^2}{(2n+1)(2n-1)}, \qquad n = 1, 2, \cdots,$$

and the remaining numbers of the theorem may be chosen as

$$b_{3n} = \frac{n}{2n+1}, \qquad n = 1, 2, \cdots,$$
$$b_{3n+1} = \frac{n+1}{2n+1}, \qquad n = 0, 1, \cdots,$$

$$b_{3n+2} = b = 1, \quad \beta = -2, \qquad n = 0, 1, \cdots.$$

That these polynomials $\{R_n(x)\}$ where

$$\begin{aligned} 2^n(1/2)_n(n!)^{-1}R_{3n}(x) &= P_n(x^3 - 2x), \\ 2^{n+1}(1/2)_{n+1}((n+1)!)^{-1}R_{3n+1}(x) &= (x^2 - 1)^{-1}(P_{n+1}(x^3 - 2x) \\ &\quad + xP_n(x^3 - 2x)), \\ 2^{n+1}(1/2)_{n+1}((n+1)!)^{-1}R_{3n+2}(x) &= (x^2 - 1)^{-1}(xP_{n+1}(x^3 - 2x) \\ &\quad + P_n(x^3 - 2x)) \end{aligned}$$

are orthogonal over a bounded domain follows from the boundedness of the coefficients in the recurrence relation (4) by a theorem due to Shohat ([**3**, p. 458]). Since the zeroes of the polynomials of the set $\{P_n(z)\}$ are dense in the domain $|z| \leqq 1$, the zeroes of the polynomials of the set $\{P_n(x^3-2x)\}$ (which are the same as the zeroes of the set $\{R_{3n}(x)\}$, $(n=0, 1, \cdots)$) are dense in the domain D consisting of the points x such that $|x^3-2x| \leqq 1$. This D is the union of the three disjoint intervals $((-1-(5)^{1/2})/2, -1)$, $((1-(5)^{1/2})/2, (-1+(5)^{1/2})/2)$, and $(1, (1+(5)^{1/2})/2)$. It follows from Szegö ([**4**, Theorem 6.1.1]) that this domain D is precisely the "true" domain of orthogonality of this set $\{R_n(x)\}$.

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