SPACES OF CONSTANCY OF CURVATURE OPERATORS

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1. Introduction. There are three kinds of Riemannian manifolds whose study is facilitated by the fact that their curvature operators have a particularly nice form; these are flat spaces, spaces of constant curvature, and spaces of constant holomorphic curvature. It is natural, therefore, on a general Riemannian manifold $M$ to study the distribution which assigns to each $m \in M$ the subspace of the tangent space $M_m$ to $M$ at $m$ on which the curvature operator behaves like one of the above types. For example, let $\mathcal{R}(m) = \{ x \in M_m \mid R_{xy} = 0 \text{ for all } y \in M_m \}$, where $R_{xy}$ denotes the curvature operator. Chern and Kuiper [1] have proved that the distribution $m \rightarrow \mathcal{R}(m)$ is integrable and its integral manifolds are flat. Maltz [4] has shown that the integral manifolds are totally geodesic and investigated their completeness properties.

The subspace $\mathcal{R}(m)$ of $M_m$ is called the space of nullity of the curvature operator at $m$. Similarly we define the space of constancy $\mathcal{S}_K(m)$ and the space of holomorphic constancy $\mathcal{K}_K(m)$ of the curvature operator at $m$. Here $\mathcal{S}_K(m)$ is the subspace of $M_m$ on which the curvature operator has constant curvature $K$, and $\mathcal{K}_K(m)$ is the subspace on which it has constant holomorphic curvature $4K$. We prove that the distributions $m \rightarrow \mathcal{R}_K(m)$ and $m \rightarrow \mathcal{K}_K(m)$ are integrable. The integral submanifolds are totally geodesic, have constant or constant holomorphic curvature and possess the same completeness properties as in the flat case. Ōtsuki [6] has also considered the spaces of constancy $\mathcal{S}_K(m)$.

2. Riemannian tensors. Let $M$ be a Riemannian manifold of class $C^\infty$; we denote by $\mathcal{F}(M)$ the ring of differentiable real-valued functions on $M$ and by $\mathcal{X}(M)$ its derivation Lie algebra, which consists of the vector fields on $M$. The metric tensor field will be denoted by $\langle \cdot, \cdot \rangle$, the Riemannian connection by $\nabla_X (X \in \mathcal{X}(M))$, and the curvature operator by $R_{XY}(X, Y \in \mathcal{X}(M))$. If $M$ is almost complex $J : \mathcal{X}(M) \rightarrow \mathcal{X}(M)$ will denote the almost complex structure. Since $\langle \cdot, \cdot \rangle$, $R$, and $J$ are tensor fields, they determine tensors on each tangent space, which we denote by the same symbols. In order to unify our proofs we consider a special class of $(1, 3)$ tensor fields.

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(2.1) **Definition.** A *Riemannian tensor field* on $M$ is a tensor field $A$ of type $(1, 3)$, which for $X, Y \in \mathfrak{X}(M)$ we regard as an $\mathcal{F}(M)$-linear map $A_{XY}: \mathcal{X}(M) \to \mathcal{X}(M)$. It is required to have the following properties:

(2.1.1) $A_{XY} = -A_{YX}$,
(2.1.2) $\langle A_{XY}(Z), W \rangle = -\langle A_{XV}(W), Z \rangle$,
(2.1.3) $\mathcal{S}A_{XY}(Z) = 0$,
(2.1.4) $\mathcal{S}\nabla_X(A)_{YZ} = 0$,

for $X, Y, Z, W \in \mathfrak{X}(M)$, where $\mathcal{S}$ denotes the cyclic sum. If $\nabla_X(A)_{YZ} = 0$ for all $X, Y, Z \in \mathfrak{X}(M)$ we say that $A$ is *parallel*, and if $\nabla_X(A) = \alpha(X)A$ for some 1-form $\alpha$ we say that $A$ is *recurrent*.

It is easy to see that (2.1.1)–(2.1.3) imply

(2.1.5) $\langle A_{XY}(Z), W \rangle = \langle A_{ZW}(X), Y \rangle$ for $X, Y, Z, W \in \mathfrak{X}(M)$.

If $\langle A_{XY}(X), Y \rangle = 0$ for all $X, Y \in \mathfrak{X}(M)$, then $A = 0$ (see Helgason [3, p. 68]). Also, it is clear that the Riemannian tensor fields are closed under addition and multiplication by real numbers.

(2.2) **Lemma.** Let $A$ be a Riemannian tensor. Then for $X, Y, Z \in \mathfrak{X}(M)$ we have

$$\mathcal{S}\{\nabla_X(A)_{YZ} - A_{[X,Y]Z}\} = 0.$$ 

**Proof.** We have $\nabla_X(Y) - \nabla_Y(X) = [X, Y]$ for $X, Y \in \mathfrak{X}(M)$, and so

$$0 = \mathcal{S}\nabla_X(A)_{YZ} = \mathcal{S}\{\nabla_X(A)_{YZ} - A_{\nabla_X(Y)Z} - A_{Y\nabla_Y(Z)}\} = \mathcal{S}\{\nabla_X(A)_{YZ} - A_{[X,Y]Z}\}.$$ 

(2.3) **Definition.** Let $m \in M$. We define

$$\alpha(m) = \{x \in M_m \mid A_{xy} = 0 \text{ for all } y \in M_m\},$$

and we denote by $\alpha$ the distribution $m \to \alpha(m)$. (Here $A_{xy}$ is the operator on $M_m$ determined by $A$.) We call $\alpha(m)$ the *space of nullity* of $A$ at $m$, $\alpha$ the *field of nullity* of $A$, and dim $\alpha(m)$ the *index of nullity* of $A$ at $m$.

(2.4) **Theorem.** Let $U$ be an open subset of $M$ on which the index of nullity of $A$ is constant. Then the distribution $\alpha$ is integrable on $U$.

**Proof.** Let $X$ and $Y$ be a vector fields in $\alpha$. From (2.2) it follows that $[X, Y]$ is in $\alpha$.

It is clear that the curvature operator is a Riemannian tensor field; however, there are two other Riemannian tensors that we shall be particularly interested in, namely $B$ and $D$, defined as follows:

$$B_{XY}(Z) = R_{XY}(Z) - K(\langle X, Z \rangle Y - \langle Y, Z \rangle X).$$
\[ D_{XY}(Z) = R_{XY}(Z) - K(\langle X, Z \rangle Y - \langle Y, Z \rangle X \\
+ \langle JX, Z \rangle JY - \langle JY, Z \rangle JX + 2\langle JX, Y \rangle JZ), \]

for \( X, Y, Z \in \mathfrak{X}(M) \), where \( K \) is a constant. The latter tensor field is defined if \( M \) is almost complex. It is not hard to verify that \( B \) is Riemannian, and if \( M \) is Kählerian, \( D \) is Riemannian. It can be shown \([3], [7]\) that \( B = 0 \) if and only if \( M \) has constant curvature \( K \), and that \( D = 0 \) if and only if \( M \) has constant holomorphic curvature \( 4K \).

We denote by \( \mathcal{B}_K \) and \( \mathcal{D}_K \) the fields of nullity of \( B \) and \( D \) respectively, and we call \( \mathcal{B}_K(m) \) and \( \mathcal{D}_K(m) \) the spaces of constancy and holomorphic constancy of the curvature operator at \( m \). We shall be concerned only with the tensor fields \( B \) and \( D \); however, there are other interesting Riemannian tensor fields. For example, if \( M \) is locally symmetric (i.e., the curvature operator is parallel), the Weyl conformal tensor field is a parallel Riemannian tensor field, and by (2.4) its field of nullity is integrable.

3. Local properties of the integral manifolds. Let \( L \) be a Riemannian manifold isometrically imbedded in another Riemannian manifold \( M \). Let \( \mathfrak{I}(L) = \{ X \mid X \in \mathfrak{X}(M) \} \); then we write \( \mathfrak{I}(L) = \mathfrak{X}(L) \oplus \mathfrak{X}(L)^\perp \) where \( \mathfrak{X}(L)^\perp \) is the collection of vector fields normal to \( L \). Let \( P : \mathfrak{I}(L) \to \mathfrak{X}(L) \) be the natural projection. For \( X, Y \in \mathfrak{X}(L) \) we denote, the Riemannian connection and curvature operator of \( L \) by \( \Gamma^X \) and \( \kappa_{XY} \) respectively. The configuration tensor \([2]\) of \( L \) in \( M \) is an \( \mathfrak{X}(M) \)-linear map \( t : \mathfrak{X}(L) \times \mathfrak{X}(L) \to \mathfrak{X}(L) \) defined by \( t_X(Y) = \nabla_X(Y) - \Delta_X(Y) \) \( (X, Y \in \mathfrak{X}(L)) \) and \( t_X(Z) = P\nabla_X(Z) \) \( (X \in \mathfrak{X}(L), Z \in \mathfrak{X}(L)^\perp) \). Then \( t_X(\mathfrak{X}(L)) \subseteq \mathfrak{X}(L)^\perp \), \( t_X(\mathfrak{X}(L)^\perp) \subseteq \mathfrak{X}(L) \) for \( X \in \mathfrak{X}(L) \), \( t_X(Y) = t_Y(X) \) for \( X, Y \in \mathfrak{X}(L) \) and \( \langle t_X(Z), W \rangle = -\langle t_X(W), Z \rangle \) for \( W, X \in \mathfrak{X}(L), Z \in \mathfrak{X}(L)^\perp \). The configuration tensor vanishes if it vanishes on either \( \mathfrak{X}(L) \) or \( \mathfrak{X}(L)^\perp \), and so it is equivalent to the second fundamental form, which in our terminology would be the map \( X \to t_X(Z) \) for \( X \in \mathfrak{I}(L), Z \in \mathfrak{X}(L)^\perp \).

We now prove that the field of nullity of a Riemannian tensor is totally geodesic. First we state a lemma, the proof of which is obvious.

(3.1) Lemma. Let \( L \) be an integral manifold of \( \alpha \). Then if \( X, Y, Z \in \mathfrak{X}(L)^\perp \), \( A_{XY}(Z) \in \mathfrak{X}(L)^\perp \).

(3.2) Theorem. Let \( L \) be an integral manifold of \( \alpha \); then \( L \) is totally geodesic.

Proof. Let \( X \in \mathfrak{X}(L) \) and \( Y, Z, U \in \mathfrak{X}(L)^\perp \). We first show that \( t_X(A_{YZ}(U)) = 0 \). Since \( A_{YZ}(U) \subseteq \mathfrak{X}(L)^\perp \) we have
\[ P \mathcal{S}_{XYZ} \nabla_X (AYZ(U)) = P \{ \nabla_X (AYZ(U)) + \nabla_Z (AXY(U)) + \nabla_Y (AZX(U)) \} = t_X (AYZ(U)). \]

On the other hand by (2.2) and (3.1)
\[ P \mathcal{S}_{XYZ} \nabla_X (AYZ(U)) = P \mathcal{S} \{ AYZ(\nabla_X (U)) + A_{[X,Y]Z}(U) \} = AYZP \nabla_X (U) + A_{P[X,Y]Z}(U) + A_{P[Z,X]Y}(U) = 0. \]

Next we let \( W \in \mathcal{K}(L) \). Then \( t_X (W) \in \mathcal{K}(L) \) and so by (3.1) \( AYZ(t_X (W)) \in \mathcal{K}(L) \); however, \( \langle AYZ(t_X (W)), U \rangle = \langle W, t_X (AYZ(U)) \rangle = 0 \) and so \( t_X (W) \in \mathcal{K}(L) \). Hence \( t_X (W) = 0 \); this proves that \( L \) is totally geodesic.

(3.3) **Corollary.** Suppose \( L \) is an integral manifold of \( \mathcal{K}_K \) or \( \mathcal{C}_K \). Then \( L \) has constant curvature \( K \) or constant holomorphic curvature \( 4K \).

If \( M \) is a Kähler manifold and \( M \) is an integral manifold of \( \mathcal{C}_K \), then \( L \) is a Kähler submanifold of \( M \).

**Proof.** The first statement follows from the Gauss equation (see [2]):
\[ PR_{XY} = r_{XY} - [t_X, t_Y]. \]

For the last statement let \( X \in \mathcal{K}(L), Y \in \mathcal{C}(L) \). Then
\[ DJ_{XY} = JD_{XY} = 0, \]
and so \( JX \in \mathcal{K}(L) \); hence \( L \) is a Kähler submanifold of \( M \).

4. **The completeness of the integral manifolds.** Let \( G \) be the set on which the index of nullity \( \mu \) assumes its minimum value \( \lambda \).

(4.1) **Proposition.** The function \( \mu \) is upper semicontinuous, and the set \( G \) is open.

**Proof.** It suffices to prove that for any \( m \in M \) there exists a neighborhood \( U \) of \( m \) such that \( \mu(p) \leq \mu(m) \) for \( p \in U \), but this is obvious.

We shall need the following lemma. Let \( n \) = dimension of \( M \).

(4.2) **Lemma.** Let \( \gamma : [0, b] \rightarrow L \) be a unit speed geodesic in an integral manifold \( L \) of \( \mathcal{A}_K \) in \( G \). Then there exists a frame field \( \{ e^1, \cdots, e^n \} \) on \( \gamma \) such that:

(4.2.1) \( e^i(t) \in \mathcal{A}(\gamma(t)) \) (1 \( \leq i \leq \lambda \), \( t \in [0, b] \)).
(4.2.2) \( \gamma'(t) = e^i(t) \) (\( i \in [0, b] \)).
(4.2.3) The frame field is parallel on \( \gamma \).
Proof. We may assume (4.2.1)-(4.2.3) hold for \( t=0 \). The frame field is defined at an arbitrary \( t \in [0, b) \) by parallel translation. It is obvious that (4.2.2) and (4.2.3) are satisfied, and (4.2.1) holds because the parallel translation takes place along the submanifold \( L \).

(4.3) Theorem. Assume \( M \) is complete and that \( A \) is recurrent. Then each integral manifold \( L \) of \( \mathcal{A} \) in \( G \) is complete.

Proof. If \( \lambda = n \) the proof is trivial, so we assume \( \lambda < n \). Let \( \gamma : [0, b) \rightarrow L \) be a unit speed geodesic. Since \( M \) is complete we may extend \( \gamma \) to a geodesic \( \gamma : [0, \infty) \rightarrow M \). Let \( \{e_1, \ldots, e_n\} \) be a frame field on \( \gamma|_{[0, b)} \) which satisfies (4.2.1)-(4.2.3). Then we may define \( \{e_1, \ldots, e_n\} \) at \( \gamma(b) \) by parallel translation; (4.2.1)-(4.2.3) now hold for \( \gamma|_{[0, b)} \). We extend each \( e_i \) to a vector field \( E_i \) defined on a neighborhood \( N \) of \( \gamma|_{[0, b]} \). Also, let \( X, Y \) be vector fields on \( N \) such that \( X_{\gamma(t)}, Y_{\gamma(t)} \subseteq \mathcal{A}(\gamma(t)) \) for \( t \in [0, b) \) and \( X, Y \) are parallel on \( \gamma|_{[0, b]} \).

Let \( \lambda + 1 \leq \rho, q, r \leq n \); we define \( \Phi_{\rho q} : [0, b] \rightarrow \mathbb{R} \) and \( \Gamma_{\rho q} : [0, b] \rightarrow \mathbb{R} \) by

\[
\Phi_{\rho q} = \langle A_{E_{\rho}E_{\rho}}(X), Y \rangle \circ \gamma
\]

By (2.1.4) and (4.2.1)-(4.2.3) it follows that

(4.3.1)

\[
\Phi_{\rho q} = 0.
\]

Since the matrix \( (\Phi_{\rho q}) \) is nonzero at 0, it follows from the theory of ordinary differential equations that \( (\Phi_{\rho q}) \) cannot vanish at \( b \). The vector fields \( X \) and \( Y \) are arbitrary and so \( \mu(\gamma(b)) = \lambda \). Therefore \( \gamma(b) \in G \) and so there exists \( c > b \) such that \( \gamma([0, c)) \subseteq G \). Hence every geodesic in \( L \) is infinitely extendable (in \( L \)) and so \( L \) is complete.

5. Some examples. (a) Consider the "dishpan surface," the graph of the function \( f : \mathbb{R}^2 \rightarrow \mathbb{R} \) defined by

\[
f(x, y) = \begin{cases} 
-\exp\left(\frac{1}{x^2 + y^2 - r^2}\right), & x^2 + y^2 < r^2, \\
0, & x^2 + y^2 \geq r^2,
\end{cases}
\]

where \( 0 \leq r < \infty \). The index of nullity \( \mu \) of the curvature operator of this surface is 0 on \( G = \{(x, y) \mid x^2 + y^2 < r^2\} \) except for one circle and 2 on the complement of \( G \). By choosing the function \( f \) differently the set \( G \) can be made to assume a variety of shapes; for example, polygonal regions or the complement of a finite set.
(b) The index of nullity of the Cartesian product of $n$ dishpan surfaces assumes the values $0, 2, \cdots, 2n$.

(c) Let $F$ be a flat manifold of dimension $k$: for example, $R^k$ or a $k$-dimensional flat torus $T^k$. If the index of nullity of a manifold $M$ assumes its minimum value $\lambda$ on $G \subset M$, then the index of nullity of the curvature operator assumes its minimum value $\lambda + k$ on $G \times F$. If $\lambda = 0$, the foliation of $G \times F$ consists of manifolds of the form \{\(p\)\} \times F (\(p \in M\)).

(d) Let $M$ be an $n$-dimensional manifold and let $G$ be $R_K$ or $S_K$, where $K \neq 0$. The author conjectures that the minimum value of the index of nullity of $G$ is either 0 or $n$.

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References


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