PERIODIC SOLUTIONS OF FOURTH-ORDER DIFFERENTIAL EQUATIONS

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The object of this paper is to prove the following result.

**Theorem.** Let $f: \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ and $g: \mathbb{R}^2 \to \mathbb{R}^2$ be $C^1$ functions and consider the differential system

$$
\begin{align*}
x' &= f(x, y), \\
y' &= g(y),
\end{align*}
$$

where $x, y \in \mathbb{R}^2$, $x' = dx/dt$ and $y' = dy/dt$. If there is a positively compact solution $(x(t), y(t))$ of (1), then (1) has a periodic solution.

Recall that a solution $(x(t), y(t))$ is said to be positively compact if $(x(t), y(t))$ remains in a compact set for all $t \geq 0$, and the solution is compact, if it remains in a compact set for all $t$ in $\mathbb{R}$.

**Proof.** First we note that we can assume that $||f(x, y)|| \leq 1$, for all $x$ and $y$, where $|| \cdot ||$ denotes a norm on $\mathbb{R}^2$. Indeed, if this were not true we could replace (1) with

$$
\begin{align*}
x' &= (1 + ||f||^{-1})f(x, y) = \tilde{f}(x, y), \\
y' &= g(y).
\end{align*}
$$

Then the solution curves of (1) agree with those of (2) and $||f|| \leq 1$.

Now let $(x(t), y(t))$ be a positively compact solution of (1). Then $y(t)$ is a positively compact solution of $y' = g(y)$ on $\mathbb{R}^2$, and by the Poincaré-Bendixson Theory, cf. [2, pp. 394–395], the positive limit set $L^+_v$ of $y(t)$ in $\mathbb{R}^2$ is nonempty and contains a periodic solution $\gamma(t)$ of $y' = g(y)$. (The solution $\gamma$ may be an equilibrium point of $y' = g(y)$.)

Now consider the second-order, periodic, differential equation

$$
x' = f(x, \gamma(t)).
$$

We claim that (3) has a compact solution. Indeed, since $(x(t), y(t))$ is a positively compact solution of (1), its positive limit set $\Omega(x, y)$ in $\mathbb{R}^4$ is nonempty, compact and invariant, cf. [4, pp. 338–340]. Let $P: \mathbb{R}^3 \times \mathbb{R}^2 \to \mathbb{R}^2$ be the mapping defined by $P: (x, y) \to y$. Then it is clear that $P(\Omega(x, y)) = L^+_v$. With $\gamma(t)$ given above, choose $x(t)$ so that $(x, \gamma) \in \Omega(x, y)$. Then $\tilde{x}(t)$ is a compact solution of (3).

Since $||f|| \leq 1$, the solutions of (3) can be continued for all time

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It then follows from Massera's Theorem [3, Theorem 2] that (3) has a periodic solution \( \dot{x}(t) \), and further, the periods of \( \dot{x}(t) \) and \( \dot{y}(t) \) agree. Consequently, \((x(t), y(t))\) is a periodic solution of (1). This completes the proof of the theorem.

REMARKS 1. It is not necessary to assume that \( f \) and \( g \) are \( C^1 \) functions. What is needed is that \( f \) and \( g \) be continuous and that the solutions of (1) be unique.

2. The domain for the \( y \)-variable can be changed to an open subset \( W \) of \( \mathbb{R}^2 \). That is, \( f \) and \( g \) are defined on \( \mathbb{R}^2 \times W \) and \( W \), respectively. Since Massera's Theorem makes use of a fixed point theorem of Brouwer, cf. [1], for mappings of \( \mathbb{R}^2 \) into \( \mathbb{R}^2 \), it does not appear that the domain of the \( x \)-variable can be similarly changed.

3. Finally, it should be noted that the above theorem admits the following generalization. Consider the equation

\[
(4) \quad z' = F(z),
\]

on \( \mathbb{R}^4 \), where \( F: \mathbb{R}^4 \to \mathbb{R}^4 \) is of class \( C^1 \). Assume that there is a \( C^1 \) diffeomorphism \( \phi: \mathbb{R}^4 \to \mathbb{R}^4 \) that changes (4) into (1). Since the solution curves of (4) are mapped onto those of (1) by \( \phi \), we can then conclude that if (4) has a positively bounded solution, it has a periodic solution.

REFERENCES


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