MAXIMUM AND MINIMUM FIRST EIGENVALUES FOR A CLASS OF ELLIPTIC OPERATORS

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Let A be an open bounded set of R^m, α a positive constant, $\alpha \leqq (1/m)$; let $\mathfrak{L}_\alpha$ be the class of elliptic differential operators

$$L \equiv \sum_{i,j}^{1,m} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j}, \tag{1}$$

with a_{ij} measurable in A and satisfying in A the conditions

$$\sum_{i,j}^{1,m} a_{ij}(x) p_i p_j \geqq \alpha \, | \, p \, |^2, \qquad \sum_{i=1}^{m} a_{ii}(x) = 1. \tag{2}$$

Observe that $(1/m)\Delta$ belongs to $\mathfrak{L}_\alpha$. Let G be the class of functions u such that u is continuous in $\bar{A}$, $u \in H^{2,m}(A)$, $u=0$ in ∂A, u positive in A. Let Λ be the set of real numbers λ for which there is an L in $\mathfrak{L}_\alpha$ and a function u in G such that

$$Lu + \lambda u = 0 \qquad \text{a.e. in } A. \tag{3}$$

By the maximum principle all the λ's are positive (see [1]). Bounds for the λ's were established under various hypotheses by Duffin [2], Protter-Weinberg [3]. Here we want to determine if Λ has a maximum or a minimum and for what operator in $\mathfrak{L}_\alpha$ the maximum or minimum occurs.

Let M_α, m_α denote the maximizing and minimizing operator relative to the class $\mathfrak{L}_\alpha$, that is for each fixed function u and fixed x:

$$M_\alpha[u(x)] = \sup_{L \in \mathfrak{L}_\alpha} Lu(x), \qquad m_\alpha[u(x)] = \inf_{L \in \mathfrak{L}_\alpha} Lu(x), \tag{4}$$

with u in G. (For definitions and general properties of such operators see [5] and, in the case of two variables, see also [4].)

THEOREM I. *Let ∂A be of class C^2. If there is a function u_1 (u_2) of class G and a constant λ' (λ'') such that*

$$M_\alpha[u_1] + \lambda' u_1 = 0 \quad \textit{a.e. in } A, \tag{5}$$

$$m_\alpha[u_2] + \lambda'' u_2 = 0 \quad \textit{a.e. in } A, \tag{6}$$

then Λ has a minimum (a maximum) and

Received by the editors February 9, 1966.

[1] This research was done in part at Louisiana State University. The author thanks Dr. A. Zettl for helpful discussions.

$$\lambda' = \min \Lambda, \qquad \lambda'' = \max \Lambda. \tag{7}$$

THEOREM II. *Let A be the sphere $\{x: |x| < 1\}$. Denote with $b(c)$ the first zero of the Bessel function J_p (J_q) where*

$$p = \frac{2(m-1)\alpha - 1}{2 - 2(m-1)\alpha}, \qquad q = \frac{1 - 2\alpha}{2\alpha}. \tag{8}$$

Then

$$\min \Lambda = [1 - (m-1)\alpha]b^2, \qquad \max \Lambda = \alpha c^2; \tag{9}$$

the equation for which the minimum occurs is

$$\sum_{i,j}^{1,m} \left[\delta_{ij} + (1 - m\alpha) \frac{x_i x_j}{|x|^2} \right] \frac{\partial^2 u}{\partial x_j \partial x_j} + [1 - (m-1)\alpha] b^2 u = 0, \tag{10}$$

and a solution of class G is

$$|x|^{-p} J_p(b|x|); \tag{11}$$

the equation for which the maximum occurs is

$$\sum_{i,j}^{1,m} \left[\frac{1-\alpha}{m-1} \delta_{ij} + \frac{m\alpha - 1}{m-1} \frac{x_i x_j}{|x|^2} \right] \frac{\partial^2 u}{\partial x_i \partial x_j} + \alpha c^2 u = 0, \tag{12}$$

and a solution of class G is

$$|x|^{-q} J_q(c|x|). \tag{13}$$

Observe that for $\alpha = 1/m$ there is only the operator $(1/m)\Delta$ in the class $\mathfrak{L}_\alpha$ and by Theorem II it follows that $\min \Lambda = \max \Lambda = m/b$ with b the first eigenvalue of $J_{(m-2)/2}$.

THEOREM III. *Let r_1 be the supremum of the radii of the spheres contained in A and r_2 the infimum of the radii of the spheres containing A. Then for any λ in Λ we have*

$$\frac{1 - (m-1)\alpha}{r_2^2} b^2 \leqq \lambda \leqq \frac{\alpha}{r_1^2} c^2,$$

where b, c are defined in Theorem II.

First we prove a lemma:

LEMMA. *Let ∂A be of class C^2 and let u in G be a solution of* (3) *for some L in $\mathfrak{L}_\alpha$ and some real λ. Denote by $d(x)$ the distance of x from ∂A; there are two positive constants k_1, k_2 such that in A*

$$k_1 d(x) \leqq u(x) \leqq k_2 d(x). \tag{14}$$

The lemma is substantially known; the proof is based on the standard use of an auxiliary function.

By hypothesis there is a constant h such that for any point x^o of ∂A two open spheres S_1 and S_2 of radius h exist such that S_1 is contained in A, S_2 has no points in common with A and x^o belongs to the closure of both S_1 and S_2. For a fixed x^o of ∂A let y be the center of S_2 and

$$g(y) = 1 - \left(\frac{h}{|x-y|}\right)^{1/\alpha}. \tag{15}$$

From (1) it follows that

$$\begin{aligned} Lg = {} & \frac{1}{2}\left(\frac{h}{|x-y|}\right)^{1/\alpha} |x-y|^{-2} \\ & \cdot \left\{1 - \left(2+\frac{1}{\alpha}\right)\sum_{i,j}^{1,m} a_{ij}\frac{(x_i-y_i)(x_j-y_j)}{|x-y|^2}\right\} \end{aligned} \tag{16}$$

and from (2)

$$Lg < -c \text{ in } A,$$

with c a positive constant depending only on α, h and the diameter of A. Denote by t the maximum of λu in $\bar{A}$; we obtain

$$L\left(\frac{t}{c}g - u\right) \leqq 0 \text{ in } A, \qquad \frac{t}{c}g - u \geqq 0 \text{ in } \partial A,$$

and by the maximum principle (see [1])

$$u \leqq \frac{t}{c}\left[1 - \left(\frac{h}{|x-y|}\right)^{1/\alpha}\right]. \tag{17}$$

For a fixed x in A denote by x^o one of the nearest points to x on ∂A; we have $|x-x^o| = d(x)$, $|x-y| = d(x)+h$ and by (17) the second inequality of (14) follows for a proper choice of k_2.

Now let y be the center of the sphere S_1 relative to a point x^o in ∂A, and let

$$T \equiv \left(x: \frac{h}{2} < |x-y| < h\right),$$

$$s = \frac{h}{2^{1/\alpha}-1} \min_{d(x)\geqq h/2} u(x);$$

it follows that

$$u + sg \geqq 0 \text{ in } \partial T$$

and by (16)

$$L(u + sg) = -\lambda u + sLg < 0 \text{ in } T,$$

then again using the maximum principle

$$u(x) \geqq s\left\{\left(\frac{h}{|x-y|}\right)^{1/\alpha} - 1\right\} \text{ in } T. \tag{18}$$

For a fixed x in A, with $d(x) < (h/2)$, denote by x^o the nearest point of ∂A; then $|x-y| = h - d(x)$ and the first inequality of (14) follows by (18); for x such that $d(x) \geqq (h/2)$ it follows by the positivity of u.

Proof of Theorem I. Suppose u_1 in G is a solution of (5).

By a previous result (see [3]) there is an operator L_1 in $\mathfrak{L}_\alpha$ such that

$$M_\alpha[u_1] = L_1 u_1.$$

Let u be a function of G which is a solution of (3), with L in $\mathfrak{L}_\alpha$. By the lemma

$$\inf_{x \in A} \frac{u_1(x)}{u(x)} = t > 0.$$

We have

$$\lambda s u - \lambda' u_1 \leqq 0 \text{ in } A \quad \text{for } s = t\frac{\lambda'}{\lambda};$$

by (4) we obtain

$$L(su - u_1) \geqq Lsu - M_\alpha[u_1] = -\lambda s u + \lambda' u_1 \geqq 0 \quad \text{a.e. in } A.$$

Also $su - u_1 = 0$ in ∂A and by the maximum principle $su - u_1 \leqq 0$ in A, that is

$$\frac{\lambda'}{\lambda} \leqq \frac{1}{t}\,\frac{u_1}{u} \text{ in } A,$$

and $\lambda' \leqq \lambda$ follows by definition of t.

A similar proof holds for the maximum of Λ; in this case we consider

$$\sup_{x \in A} \frac{u_2(x)}{u(x)},$$

which is finite by the lemma.

Proof of Theorem II. By (8) and (2) it follows that p and $q > -1$.

Let r be the distance from the origin and $\phi(r)$ the function given by (13). By elementary properties of Bessel functions

$$\phi(r) = \sum_{i=0}^{\infty} (-1)^i \left(\frac{c}{2}\right)^{2i+q} \frac{r^{2i}}{i!\Gamma(i+q+1)},$$

and ϕ is nonnegative, decreasing and of class C^2 in [0,1],

$$\phi'(0) = 0, \qquad \phi(1) = 0; \tag{19}$$

furthermore ϕ is a solution of the equation

$$\phi'' + \frac{1+2q}{r}\phi' + c^2\phi = 0,$$

that is

$$\alpha\phi'' + \frac{1-\alpha}{r}\phi' + c^2\alpha\phi = 0. \tag{20}$$

We prove that

$$\phi'' - \frac{\phi'}{r} \geqq 0 \text{ in } (0, 1]. \tag{21}$$

By (20)

$$\phi'' - \frac{\phi'}{r} = -\frac{1}{\alpha}\frac{\phi'}{r} - c^2\phi. \tag{22}$$

Inequality (21) holds in the limit at $r=0$ and, by (22) it holds for $r=1$. If (21) does not hold

$$-\frac{1}{\alpha}\frac{\phi'}{r} - c^2\phi \tag{23}$$

has a minimum in (0, 1) and its derivative is zero at this point, that is

$$-\frac{1}{\alpha}\frac{\phi''}{r} + \frac{1}{\alpha}\frac{\phi'}{r^2} - c^2\phi' = 0,$$

and by (20) it follows that at the point of minimum (23) is positive.

Let $u(x) = \phi(r)$; u belongs to G. Let $C_i[u]$ be the principal curvatures of u, i.e. the eigenvalues of the matrix $|(\partial^2 u)/(\partial x_i \partial x_j)|$ ordered in the following way

$$C_1[u] \leqq C_2[u] \leqq \cdots \leqq C_m[u].$$

By a previous result, see [3],

$$m_\alpha[u] = [1 - (m-1)\alpha]C_1[u] + \alpha \sum_{i=2}^{m} C_i[u].$$

By (21)

$$(24) \qquad C_m[u] = \phi'', \qquad C_i[u] = \frac{\phi'}{r}, \qquad i = 1, 2, \cdots, m-1,$$

then by (20)

$$m_\alpha[u] + c^2 u = 0 \text{ in } A.$$

The second equality of (9) follows by Theorem I. Since

$$\frac{\partial^2 u}{\partial x_i \partial x_j} = \phi'' \frac{x_i x_j}{r^2} + \phi'\left(\frac{\delta_{ij}}{r} - \frac{x_i x_j}{r^3}\right),$$

(12) follows.

The function $\phi(r)$ defined by (11) is a solution of

$$[1 - (m-1)\alpha]\phi'' + \frac{(m-1)\alpha}{r}\phi' + [1 - (m-1)\alpha]b^2\phi = 0.$$

By the same argument (21) holds and hence also (24) with $u(x) = \phi(r)$. Similarly, using

$$M_\alpha[u] = [1 - (m-1)\alpha]C_m[u] + \alpha \sum_{i=1}^{m-1} C_i[u],$$

we complete the proof.

Proof of Theorem III. Let r be the radius of a sphere S, with $\overline{A} \subset S$. By Theorem II there follows the existence of a function v, $v \in H^{2,m}(S) \cap C^0(\overline{S})$, such that

$$M_\alpha v + \frac{\alpha c^2}{r^2} v = 0, \qquad v > 0 \text{ in } S,$$

$$v = 0 \text{ in } \partial S.$$

Let

$$t = \max_{x \in \overline{A}} \frac{u(x)}{v(x)}$$

where u is a function of class G and a solution of (3) with $L \in \mathfrak{L}_\alpha$.

The nonpositive function $u - tv$ takes its maximum inside A and at that point $L(u - tv) \leqq 0$; but

$$L(u - tv) \geqq -\lambda u + t\frac{\alpha c^2}{r^2}v = \lambda(tv - u) + tv\left(\frac{\alpha c^2}{r^2} - \lambda\right) \text{ in } A.$$

Hence

$$\frac{\alpha c^2}{r^2} - \lambda \leqq 0.$$

Observation I. In Theorem I the hypothesis "Let ∂A be of class C^2" can be replaced by "Let A be star-shaped." The hypothesis of smoothness of the boundary was used only to prove that

$$\inf_{x \in A} \frac{u_1(x)}{u(x)} > 0. \tag{25}$$

Using a dilatation we can replace u_1 by a function v such that

$$v > 0 \text{ in } \bar{A}, \qquad M_\alpha v + \bar{\lambda} v = 0 \text{ in } A,$$

with $\bar{\lambda}$ as near as we like to λ'. By the argument of the proof of Theorem I it follows that $\lambda \geqq \bar{\lambda}$ and then $\lambda \geqq \lambda'$.

A similar argument holds for the maximum of Λ.

Observation II. In Theorem I the hypothesis "Let ∂A be of class C^2" can be replaced by an hypothesis used by Duffin in a similar problem [2]: "We suppose that a function w of class $C^2(\bar{A})$ exists such that

$$w > 1 \text{ in } A, \qquad M_\alpha w + kw < 0 \text{ in } \partial A,$$

where $k = \sup \Lambda$."

We observe that if (25) holds the proof is the same as before. If (25) does not hold, for ϵ positive and sufficiently small

$$\frac{u_1 + \epsilon w}{u}$$

takes its minimum in a point so near to the boundary that there $M_\alpha w + kw < 0$. In this point of minimum we must have

$$L\frac{u_1 + \epsilon w}{u} \geqq 0,$$

and by the hypothesis and some computations it follows that $\lambda' \leqq \lambda$. The other part of the proof is obtained considering

$$\frac{u_2}{u + \epsilon w}.$$

Observation III. Let β, γ denote two nonnegative constants; the previous considerations can be extended to the class $\mathfrak{L}_{\alpha,\beta,\gamma}$ of elliptic operators L

$$L = L_1 + \sum_{i=1}^{m} b_i \frac{\partial}{\partial x_i} + c,$$

with b_i, c measurable in A and

$$L_1 \in \mathfrak{L}_\alpha, \qquad \sum_{i=1}^{m} b_i^2 \leqq \beta^2, \qquad -\gamma \leqq c \leqq 0.$$

The maximizing and minimizing operator related to the class $\mathfrak{L}_{\alpha,\beta,\gamma}$ is studied in [5].

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