Since \( g_2 = 0, g_3 = 1 \), it is easy to prove inductively that \( C_n = 0 \) unless \( n \equiv 0 \) modulo 3. Substitution in (2) shows that \( zf(z) \) is a function of \( z^3 \), so that

\[
f(\eta z) = \eta^2 f(z), \quad \eta^3 = 1.
\]

This shows that \( f(-\eta^2 h(z)) = \eta f(-h(z)) \), and the proof of the equivalence of the various expressions for \( G \) in (5) is complete.

References


Imperial College of Science and Technology, London

ON THE BOUNDARY BEHAVIOR OF FUNCTIONS MEROMORPHIC IN THE UNIT DISK

PETER COLWELL

1. Introduction. Let \( f(z) \) be meromorphic in \( D : \{ |z| < 1 \} \), and suppose that the values assumed by \( f(z) \) in \( D \) lie in a domain \( G \) whose boundary \( \Gamma \) has positive logarithmic capacity. Then \( f(z) \) is of bounded characteristic in \( D \) and has finite radial limits \( f(e^{i\theta}) = \lim_{r \to 1} f(re^{i\theta}) \) at almost all points \( e^{i\theta} \) on \( C : \{ |z| = 1 \} \). (For this and more general theory of meromorphic functions, see \([4, \text{pp. 208 ff.}]\).) The class of functions satisfying these conditions and having the additional property that \( f(e^{i\theta}) \) belongs to \( \Gamma \) almost everywhere on \( C \) has been studied by O. Lehto \([3]\) and D. A. Storvick \([6]\), who called it class \((L)\).

If \( A \) is a sequence of points in \( D \) satisfying \( \sum_{a \in A} (1 - |a|) < \infty \), the Blaschke product with respect to \( A \) in \( D \) is the function \( B(z; A) = \prod_{a \in A} \left[ 1 - a \right] \). The present note arises from a suggestion by Professor Storvick that the following theorem, established in \([1]\), be extended to functions in class \((L)\). Here we denote by \( A' \) the derived set of \( A \).

Theorem 1. Let \( E \) be a set on \( C \). A necessary and sufficient condition

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that there exist a Blaschke product \( B(z; A) \) for which \( B(e^{i\theta}) = \lim_{r \to 1} B(re^{i\theta}; A) \) is defined and of modulus one at every point of \( C \) and such that \( A' = E \) is that \( E \) be closed and nowhere dense on \( C \).

2. Let \( G \) be a domain whose boundary \( \Gamma \) has positive logarithmic capacity.

**Theorem 2.** Let \( f(z) \) be a function of class \((L)\) with respect to \( G \) and \( \Gamma \), and suppose that \( f(e^{i\theta}) = \lim_{r \to 1} f(re^{i\theta}) \) exists and belongs to \( \Gamma \) for every \( e^{i\theta} \) on \( C \). Then if \( a \) is any point of \( G \) and \( A = \{ z \in D : f(z) = a \} \), \( A' \) is closed and nowhere dense on \( C \).

We note first that \( A \) is not empty, since O. Lehto [3, p. 12] showed that any omitted value of \( f \) in \( G \) is a radial limit for \( f(z) \). Also from [3, p. 12], if \( A \) is finite then \( a \) must be a boundary value for \( f(z) \), unless \( G \) is simply-connected and \( f(z) = \rho \{ R(z) \} \) for \( R(z) \) rational, 
\[ |R(z)| \leq 1, \quad |R(e^{i\theta})| = 1, \quad \text{and} \quad w = \rho(z) \text{ a schlicht function mapping } D \text{ onto } G. \]
In the latter case \( f(z) \) assumes each of its values only finitely many times, and the theorem is clearly true.

For the case when \( A \) is infinite, \( A' \cap D = \emptyset \) since \( f(z) \) is meromorphic in \( D \), and \( A' \) is closed. For any point \( e^{i\theta} \) of \( A' \), \( f(e^{i\theta}) = \lim_{r \to 1} f(re^{i\theta}) \) is a point \( \gamma(\theta) \) of \( \Gamma \). That is, the radial cluster set, \( C_r(f, e^{i\theta}) \), for \( f \) at \( e^{i\theta} \) is a single point \( \gamma(\theta) \). However, the interior cluster set, \( C_D(f, e^{i\theta}) \), for \( f \) at \( e^{i\theta} \) contains at least the points \( a \) and \( \gamma(\theta) \). Thus for \( e^{i\theta} \) in \( A' \) we have \( C_D(f, e^{i\theta}) \neq C_r(f, e^{i\theta}) \). By a theorem of E. F. Collingwood [2, p. 378], \( A' \) must be a set of category \( I \) on \( C \). Since \( A' \) is closed, \( A' \) is necessarily nowhere dense on \( C \). (The method of proof that \( A' \) is nowhere dense on \( C \) was originally suggested to the author by Professor K. Noshiro for use in the proof of Theorem 1.)

3. In the special case that \( G \) is simply-connected and its boundary \( \Gamma \) is a Jordan curve, we prove the following

**Theorem 3.** Let \( a \) be any point of a simply-connected domain \( G \) whose boundary \( \Gamma \) is a Jordan curve. Let \( E \) be a closed nowhere dense set on \( C \). Then there exists a function, analytic in \( D \), such that: (i) \( f(z) \) assumes its values in \( G \); (ii) for every \( e^{i\theta} \) on \( C \) the limit \( \lim_{r \to 1} f(re^{i\theta}) = f(e^{i\theta}) \) exists and belongs to \( \Gamma \); (iii) if \( A = \{ z \in D : f(z) = a \} \), then \( A' = E \).

If \( \rho = g(w) \) is a conformal mapping of \( G \) onto \( \{ \| \rho \| < 1 \} \) with \( g(a) = 0, \ g'(a) > 0 \), then \( g \) can be extended to a homeomorphism of \( G \cup \Gamma \) onto \( \{ \| \rho \| \leq 1 \} \), and we can consider the mapping \( w = g^{-1}(\rho) \) as a homeomorphism of \( \{ \| \rho \| \leq 1 \} \) onto \( G \cup \Gamma \) which is analytic in \( \{ \| \rho \| < 1 \} \) and assumes values there in \( G \).

By Theorem 1, since \( E \) is closed and nowhere dense on \( C \), there
exists a Blaschke product \( B(z; H) \) in \( D \) with radial limits of modulus one at every point of \( C \) such that \( H' = E \). (Here \( H \) is the set of zeros for \( B(z; H) \) in \( D \), and \( \sum_{h \in H} (1 - |h|) < \infty \).) We let \( \rho = B(z; H) \) and define \( w = g^{-1}[B(z; H)] \) for \( z \) in \( D \).

The image under \( B(z; H) \) of \( D \) is the disk \( \{ |\rho| < 1 \} \), since W. Seidel [5] showed that any omitted value in \( \{ |\rho| < 1 \} \) is a radial limit value for \( B(z; H) \) at some point of \( C \). It is easily verified that \( w = f(z) \) is analytic in \( D \) and satisfies (i), (ii), and (iii).

**Bibliography**


Iowa State University