ON CONFORMALLY-FLAT RIEMANNIAN
SPACE OF CLASS ONE

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1. The purpose of this paper is two-fold; first, to obtain necessary and sufficient conditions that a conformally-flat orientable Riemannian space \( C^n \) with \( n \geq 3 \) be of class one; second, to obtain a normal form for the metric of such a space. A Riemannian space \( V_n \) is a conformally-flat space if there exists a scalar function \( \sigma \) such that the product \( \sigma g_{ij} \) of \( \sigma \) and the fundamental tensor \( g_{ij} \) has zero curvature; it is of class one if it is isometrically embeddable as a hypersurface in a Euclidean space. The conformal flatness property can be expressed by the condition that \( s_i = \frac{1}{2} \partial_i \log \sigma \) is related to the curvature tensor by

\[
R_{hijk} + g_{hk}s_{ij} + g_{ij}s_{hk} - g_{hjs}g_{ik} - g_{ik}s_{hj} = 0,
\]

where

\[
s_{ij} = \nabla_i s_j - s_i \delta_j + \frac{1}{2} g_{ijs} s_k^k.
\]

The condition of class one, for an orientable space, implies the existence of a (second fundamental) symmetric tensor \( b_{ij} \) such that

\[
R_{hijk} = b_{hj}b_{ik} - b_{hk}b_{ij}; \quad \nabla_i b_{jk} - \nabla_j b_{ik}.
\]

The converse is true in the local sense.

The algebraic relations (1), (3) lead to a result of J. A. Schouten [1] which states that \( n - 1 \) of the eigenvalues of \( b_{ij} \) at each point of a \( C^n \) are equal. Denote this value by \( \rho \), the remaining eigenvalue by \( \bar{\rho} \) and denote by \( e_i \) the eigenvector of \( b_{ij} \) belonging to \( \bar{\rho} \). The quantities \( \rho, \bar{\rho} \) are also known as the principal normal curvatures and \( e_i \) the unit vector tangential to the line of curvature corresponding to \( \bar{\rho} \). Assume that \( \bar{\rho} \neq \rho \neq 0 \). Then

\[
b_{ij} = \rho g_{ij} + (\bar{\rho} - \rho) e_i e_j
\]

and by contraction of (3) we express the Ricci tensor in terms of \( g_{ij} \) and \( e_i e_j \); or in \( g_{ij} \) and \( b_{ij} \). We thus find (a), (b) below; by the second identity in (3) together with the property of conformal flatness we find (c) below.

\[
(b) \quad b_{ij} = -\frac{1}{n-2} \left( \frac{1}{\rho} R_{ij} + \bar{\rho} g_{ij} \right).
\]

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(b) \( R_{hijk} = \rho^2(g_{hj}g_{ik} - g_{hk}g_{ij}) \\
+ \rho(\bar{\rho} - \rho)(g_{hj}e_i e_k + g_{ik}e_h e_j - g_{hk}e_i e_j - g_{ij}e_h e_k), \)
\( \partial_i \rho \) is proportional to \( e_i \).

These formulas are due to Verbickii [2]; he also showed that the existence of scalar functions \( \rho, \zeta, \bar{\rho} \) and a unit vector field \( e_i \) such that (b), (c) hold is sufficient that \( V_n \) be locally a \( C^1_n \).

2. The above results are most easily verified by choosing an orthonormal basis for the tangent space consisting of eigenvectors of \( b_{ij} \); we choose \( e_i \) to be the first of these. Then \( g_{ij} \) and \( b_{ij} \) take diagonal forms with respect to this basis,
\[
[g_{ij}] = \text{diag}(1, 1, \cdots), \quad [b_{ij}] = \text{diag}(\rho, \rho, \rho, \cdots).
\]
For brevity we only give the first two diagonal elements:
\[
[g_{ij}] = \text{diag}(1, 1); \quad [b_{ij}] = \text{diag}(\bar{\rho}, \rho, \rho, \cdots); \quad [e_i e_j] = \text{diag}(1, 0).
\]

Then
\[
[R_{ij}] = \text{diag}(-(n - 1)\rho \bar{\rho}, -(n - 2)\rho^2 + \rho \bar{\rho});
\]
and among \( g_{ij}, b_{ij}, R_{ij}, e_i e_j \) any one can be written as a linear combination of any two. This is how (5) below is proved.

**Theorem 1.** If a \( V_n \) is a \( C^1_n \), then there are scalars \( E \neq 0 \) and \( F \) such that
\[
(5) \quad R_{hijk} = E(R_{hj}R_{ik} - R_{hk}R_{ij}) + F(g_{hj}g_{ik} - g_{hk}g_{ij}).
\]
Conversely, if in a \( C_n \) scalars \( E \neq 0 \), \( F \) exist such that (5) holds, where
\[
(6) \quad R = -\frac{n - 1}{(n - 2)E} + (n - 1)(n - 2)F,
\]
then \( C_n \) is a \( C^1_n \).

**Proof of the converse.** Contraction of (5) with \( g^{hk} \) gives
\[
R_{ij} = ER_{ij}R_{ik} - ERR_{ij} - (n - 1)F g_{ij}.
\]
Hence, every eigenvalue \( \lambda \) of \( R_{ij} \) satisfies
\[
\lambda = E \lambda^2 - ER \lambda - (n - 1)F;
\]
which, by (6), has as its solutions
\[
\lambda = \frac{-1}{(n - 2)E}, \quad \tilde{\lambda} = (n - 1)(n - 2)F.
\]
By (6), \( \lambda \) has multiplicity \( n - 1 \); \( \tilde{\lambda} \) has multiplicity 1. The situation is now easily reduced to that of a \( C_n \) involving a second fundamental tensor \( b_{ij} \) which is a linear combination of \( g_{ij} \) and \( e_i e_j \), where \( e_i \) is a unit eigenvector of \( R_{ij} \) associated with \( \lambda \). It is a simple exercise to relate the \( \lambda, \tilde{\lambda} \) above with \( \rho, \tilde{\rho} \) resulting in

\[
\lambda = - \left\{ (n - 2)\rho^2 + \rho \tilde{\rho} \right\}, \quad \tilde{\lambda} = -(n - 1)\rho \tilde{\rho}.
\]

We thus obtain

**Theorem 2.** If a \( V_n \) is a \( C_n^1 \), then

\[
R_{hijk} = \frac{R_{hjR_{ik}} - R_{hkR_{ij}}}{(n - 2)\left\{ (n - 2)\rho^2 + \rho \tilde{\rho} \right\}} - \frac{\rho \tilde{\rho}}{n - 2} (g_{hj}g_{ik} - g_{hk}g_{ij}),
\]

where \( \rho \neq 0, \tilde{\rho} \neq 0 \) are scalars. Conversely if a \( C_n \) satisfies (7), then \( C_n \) is a \( C_n^1 \) if \( R = - (n - 1)\left\{ (n - 2)\rho^2 + 2\rho \tilde{\rho} \right\} \).

3. Theorem 1 of §2 can be applied to find the metric of a \( C_n^1 \). This can be done by taking the fundamental tensor of a \( C_n \) in the form \( g_{ii} = 1/\phi^2, \ g_{ij} = 0, \ (i \neq j) \), and looking for the general form of \( \phi \) for which the equations (5) and (6) are satisfied. The fundamental tensor is then obtained in a canonical form as

\[
g_{ii} = 1/[f(U)]^2, \quad g_{ij} = 0, \quad (i \neq j), \quad \text{where} \quad U = \sum_i (X^i)^2 + c
\]

and \( X^i = ax^i + b^i \) with \( a \neq 0, b, c \) constants,

where \( f \) is any real analytical function of \( U \) subject to a restriction stated below. The normal form of the metric is now obtained by taking \( a = 1, b^i = c = 0 \) in (8).

This metric and some properties which have been obtained in previous papers [3], [4] are stated in the following theorem:

**Theorem 3.** The coordinates of any \( C_n^1 \) may be so chosen that its metric assumes the normal form

\[
ds^2 = \sum_i (dx^i)^2/[f(\theta)]^2, \quad \theta = \sum_i (x^i)^2,
\]

where \( f \) is any real analytic function of \( \theta \) subject to the restriction

\[(n - 1)ff' + \theta ff'' - (n - 1)\theta f'^2 \neq 0, \quad (f' = df/d\theta, \ etc.).\]

If \( \rho \neq 0 \) and \( \tilde{\rho} \) are the eigenvalues of multiplicity \( n - 1 \) and 1 respectively of the second fundamental tensor of the space (9), then

\[
\rho^2 = 4f'(f - \theta f'), \quad \rho \tilde{\rho} = 4(ff' + \theta ff'' - \theta f'^2).
\]
The eigenvector $e_i = x^i/\theta^{1/2}f$ corresponding to $\bar{p}$ is orthogonal to the hypersurface having constant curvature $\tilde{k}^2 = f^2/\theta$. If the $C^a_n$ is symmetric in the sense of Cartan, then either $f = a\theta + b$ (a space of constant curvature) or $f = c\theta^{1/2}$, where $a$, $b$, $c$ are nonzero constants. In the second case $e_i$ is a parallel vector field and the $C^a_n$ is reducible.

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References


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