CORRECTION TO "ON MATRICES WHOSE REAL LINEAR COMBINATIONS ARE NONSINGULAR"

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We are grateful to Professor B. Eckmann for pointing out an error in the proof of Lemma 4(b) of our paper [1]. This error invalidates Lemma 4(b) and that part of Theorem 1 which states the values of \( Q(n) \), \( Q_H(n) \). The error occurs immediately after the words "arguing as is usual for the complex case, we find"; it consists in manipulating as if the ground field \( \Lambda \) were commutative.

The proof of Lemma 4(b) can be repaired, as will be shown below, but it leads to a different conclusion from that given. Our paper should therefore be corrected as follows.

(i) In Theorem 1, the values of \( Q(n) \) and \( Q_H(n) \) should read

\[
Q(n) = \rho(\frac{1}{2}n) + 4, \quad Q_H(n) = \rho(\frac{1}{2}n) + 5.
\]

The two sentence paragraph following Theorem 1 should be deleted. It remains interesting to ask what topological phenomena (if any) can be related to our algebraic results.

(ii) In Lemma 4, part (b) should read

\[
Q_H(n) + 3 \leq R(4n).
\]

The proof is as follows.

Let \( W \) be a \( k \)-dimensional space of \( n \times n \) Hermitian matrices with entries from \( Q \) which has the property \( P \). The space \( Q^n \) is a real vector space of dimension \( 4n \). For each \( A \in W \) and each pure imaginary \( \mu \in Q \) we consider the following real-linear transformation from \( Q^n \) to itself:

Received by the editors December 20, 1965.
$B(x) = Ax + x\mu$.

We claim that the $(k+3)$-dimensional space formed by such $B$ has the property $P$. For suppose that such a $B$ is singular; then there is a nonzero $x$ such that

$$Ax = -x\mu;$$

then we have

$$x^*(Ax) = -x^*x\mu,$$

$$(x^*A)x = (-x\mu)^*x = \mu x^*x.$$  

Since $x^*x$ is real and nonzero, we have $\mu = 0$; hence $A$ is singular and $A=0$. This completes the proof.

(iii) In Lemma 5, there should be added a second part, reading

"(b) $R_H(n) + 3 \leq Q(n)$.”

PROOF. Let $W$ be a $k$-dimensional space of $n \times n$ real symmetric matrices which has the property $P$. Consider the matrices

$$A + \mu I,$$

where $A$ runs over $W$ and $\mu$ runs over the pure imaginary elements of $Q$. We claim that they form a space of dimension $k+3$ with the property $P$. In fact, suppose that such a matrix is singular; and suppose to begin with, that $\mu$ is nonzero. Then the elements $1$, $\mu$ form an $R$-base for a subalgebra of $Q$ which we may identify with $C$. Choose a $C$-base of $Q$; this splits $Q^n$ as the direct sum of two copies of $C^n$. Since the matrix $A + \mu I$ acts on each summand, it must be singular on at least one. That is, the real symmetric matrix $A$ has a nonzero complex eigenvalue which is purely imaginary, a contradiction. Hence $\mu$ must be zero and $B = A$. Now choose an $R$-base of $Q$; this splits $Q^n$ as the direct sum of 4 copies of $R^n$. Since $A$ acts on each summand, it must be singular on at least one. That is, $A$ must be singular; hence $A=0$. This completes the proof.

(iv) The final paragraph of the paper should be deleted, and replaced by the following proof.

"Finally, Lemmas 5(b), 3 and 4(b) show that

$$Q(n) - R_H(n) \geq 3,$$

$$Q_H(2n) - Q(n) \geq 1,$$

$$R(8n) - Q_H(2n) \geq 3.$$  

But we have already shown that
\[ R(8n) - R_H(n) = 7, \]
so all these inequalities are equalities. This completes the proof of Theorem 1."

We note that this method provides an alternative proof of Lemma 5
\[ (R(8n) - R_H(n) \geq 7), \]
without using the Cayley numbers.

**BIBLIOGRAPHY**


**Manchester University, Manchester, England**

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