CORRECTION TO "ON MATRICES WHOSE REAL LINEAR COMBINATIONS ARE NONSINGULAR"

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We are grateful to Professor B. Eckmann for pointing out an error in the proof of Lemma 4(b) of our paper [1]. This error invalidates Lemma 4(b) and that part of Theorem 1 which states the values of \(Q(n), Q_H(n)\). The error occurs immediately after the words “arguing as is usual for the complex case, we find”; it consists in manipulating as if the ground field \(\Lambda\) were commutative.

The proof of Lemma 4(b) can be repaired, as will be shown below, but it leads to a different conclusion from that given. Our paper should therefore be corrected as follows.

(i) In Theorem 1, the values of \(Q(n)\) and \(Q_H(n)\) should read

\[
Q(n) = \rho(\frac{1}{2}n) + 4, \quad Q_H(n) = \rho(\frac{1}{2}n) + 5.
\]

(ii) In Lemma 4, part (b) should read

\[
Q_H(n) + 3 \leq R(4n).
\]

The proof is as follows.

Let \(W\) be a \(k\)-dimensional space of \(n\times n\) Hermitian matrices with entries from \(Q\) which has the property \(P\). The space \(Q^n\) is a real vector space of dimension \(4n\). For each \(A \in W\) and each pure imaginary \(\mu \in Q\) we consider the following real-linear transformation from \(Q^n\) to itself:

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We claim that the \((k+3)\)-dimensional space formed by such \(B\) has the property \(P\). For suppose that such a \(B\) is singular; then there is a nonzero \(x\) such that

\[
Ax = -x\mu;
\]

then we have

\[
x^*(Ax) = -x^*x\mu,
\]

\[
(x^*A)x = (-x\mu)^*x = \mu x^*x.
\]

Since \(x^*x\) is real and nonzero, we have \(\mu = 0\); hence \(A\) is singular and \(A = 0\). This completes the proof.

(iii) In Lemma 5, there should be added a second part, reading

"(b) \(\mathcal{R}_H(n) + 3 \leq \mathcal{Q}(n)\)."

Proof. Let \(W\) be a \(k\)-dimensional space of \(n \times n\) real symmetric matrices which has the property \(P\). Consider the matrices

\[
A + \mu I,
\]

where \(A\) runs over \(W\) and \(\mu\) runs over the pure imaginary elements of \(Q\). We claim that they form a space of dimension \(k+3\) with the property \(P\). In fact, suppose that such a matrix is singular; and suppose to begin with, that \(\mu\) is nonzero. Then the elements \(1, \mu\) form an \(R\)-base for a subalgebra of \(Q\) which we may identify with \(C\). Choose a \(C\)-base of \(Q\); this splits \(Q^n\) as the direct sum of two copies of \(C^n\). Since the matrix \(A + \mu I\) acts on each summand, it must be singular on at least one. That is, the real symmetric matrix \(A\) has a nonzero complex eigenvalue which is purely imaginary, a contradiction. Hence \(\mu\) must be zero and \(B = A\). Now choose an \(R\)-base of \(Q\); this splits \(Q^n\) as the direct sum of 4 copies of \(R^n\). Sinc \(A\) acts on each summand, it must be singular on at least one. That is, \(A\) must be singular; hence \(A = 0\). This completes the proof.

(iv) The final paragraph of the paper should be deleted, and replaced by the following proof.

"Finally, Lemmas 5(b), 3 and 4(b) show that

\[
\mathcal{Q}(n) - \mathcal{R}_H(n) \geq 3,
\]

\[
\mathcal{Q}_H(2n) - \mathcal{Q}(n) \geq 1,
\]

\[
\mathcal{R}(8n) - \mathcal{Q}_H(2n) \geq 3.
\]

But we have already shown that
$R(8n) - R_H(n) = 7$, so all these inequalities are equalities. This completes the proof of Theorem 1.

We note that this method provides an alternative proof of Lemma 5 ($R(8n) - R_H(n) \geq 7$), without using the Cayley numbers.

**BIBLIOGRAPHY**


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