A NOTE ON THE RIEMANN ZETA FUNCTION

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The object of this note is to add one more formula to the vast literature on the Riemann zeta function.[1] The formula does not appear to have any immediate application but is interesting in that it is a kind of interpolation formula.[2] Specifically we shall prove

THEOREM. *If s is not an integer $s=\sigma+it$ then*

$$\frac{\pi\zeta(s)}{\sin \pi s} = \sum_{k=2}^{\infty} \frac{(-1)^k\zeta(k)}{k(k-s)} + \frac{1-\gamma}{s-1} - \frac{1}{(s-1)^2} - \frac{1}{2s^2} - \frac{1}{2s}\log 2\pi$$
$$+ \sum_{k=1}^{M} \frac{\zeta(1-2k)}{1-2k}\,\frac{1}{s-(1-2k)} + \int_1^{\infty} H_M(z) z^{-s-1}\,dz,$$

where

$$H_M(z) = O\left(\frac{B_{M+1}}{(M+1)(2M+1)}\quad\frac{1}{2M+\sigma+1}\right),$$

where γ is Euler's constant and B_{M+1} is the $M+1$st Bernoulli number. The integral converges for $\sigma > -M$.

PROOF. The proof is quite simple and involves no complexity. We start with

$$\frac{1}{\Gamma(z)} = ze^{\gamma z}\prod_1^{\infty}\left(1+\frac{z}{n}\right)e^{-z/n}.$$

If $|z|<1$, then

$$\log\frac{e^{-\gamma z}}{\Gamma(z+1)} = \sum_{n=1}^{\infty}\left(\log\left(1+\frac{z}{n}\right)-\frac{z}{n}\right) \tag{1}$$
$$= \sum_{n=1}^{\infty}\sum_{k=2}^{\infty}\frac{(-1)^{k-1}z^k}{kn^k}$$
$$= \sum_{k=2}^{\infty}\frac{(-1)^{k-1}z^k}{k}\zeta(k).$$

Using the calculus of residues, it can readily be shown that the right hand side of (1) is

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[1] The author is unable to find any similar formula in the literature.

[2] Compare with the so-called Cardinal series.

$$\frac{1}{2\pi i}\int_{3/2-i\infty}^{3/2+i\infty}\frac{\pi z^s\zeta(s)}{s\sin\pi s}\,ds,$$

provided that $|\arg z|\leqq\pi-\delta,\ \delta>0$. Consequently

$$\log\frac{e^{-\gamma z}}{\Gamma(1+1/z)}=\frac{1}{2\pi i}\int_{3/2-i\infty}^{3/2+i\infty}\frac{\pi z^{-s}\zeta(s)}{s\sin\pi s}\,ds,\tag{2}$$

subject to the above conditions.

If $s=\sigma+it$ is not an integer and $\sigma>1$, we can apply a suitable form of the Mellin transform to get

$$\begin{aligned}\frac{\pi\zeta(s)}{s\sin\pi s}&=\lim_{\lambda\to\infty}\int_{1/\lambda}^{\lambda}\log\frac{e^{-\gamma z}}{\Gamma(1+z^{-1})}z^{s-1}\,dz\\&=\lim_{\lambda\to\infty}\left(\int_{1/\lambda}^{1}+\int_{1}^{\lambda}\right)\log\frac{e^{-\gamma z}}{\Gamma(z^{-1}+1)}z^{s-1}\,dz\\&=A+B.\end{aligned}\tag{3}$$

On the other hand, it is easily verified that

$$\begin{aligned}A&=\lim_{\lambda\to\infty}\sum_{k=2}^{\infty}\frac{(-1)^k}{k}\zeta(k)\int_{1/\lambda}^{1}z^{k-s-1}\,dz\\&=\sum_{k=2}^{\infty}\frac{(-1)^k}{k(k-s)}\zeta(k),\end{aligned}\tag{4}$$

the interchange of limiting processes presenting no difficulty. Consequently since B converges for $1<\sigma<2$, we conclude that

$$\frac{\pi\zeta(s)}{s\sin\pi s}=\sum_{k=2}^{\infty}\frac{(-1)^k}{k(k-s)}\zeta(k)+\int_{1}^{\infty}\log\frac{e^{-\gamma z}}{\Gamma(1+z)}z^{-s-1}\,dz.\tag{5}$$

The integral on the right however is

$$\frac{-\gamma}{s-1}-\frac{1}{s^2}-\int_{1}^{\infty}\log\Gamma(z)z^{-s-1}\,dz.\tag{6}$$

We apply Stirling's formula to $\log\Gamma(z)$

$$\begin{aligned}\log\Gamma(z)&=(z-\tfrac{1}{2})\log z-z+\tfrac{1}{2}\log 2\pi\\&\quad+\sum_{r=1}^{M}\frac{(-1)^{r-1}B_r}{2r(2r-1)}z^{-2r+1}+H_M(z)\\&=h(z)+\sum_{r=1}^{M}\frac{(-1)^{r-1}B_r}{2r(2r-1)}z^{-2r+1}+H_M(z),\end{aligned}$$

where $h(z)=(z-\frac{1}{2})\log z-z+\frac{1}{2}\log 2\pi$ and

$$H_M(z) = O\left(\frac{B_{M+1}}{(M+1)(2M+1)}\,\frac{1}{|z|}\,2M+1\right).$$

$\int_1^\infty h(z)z^{-s-1}dz$ involves only elementary factors which can be integrated out. Having done so and combining with (6), we see that

$$B = \frac{1-\gamma}{s-1} - \frac{1}{(s-1)^2} - \frac{1}{2s^2} - \frac{\log 2\pi}{2s} + \sum_{r=1}^{M} \frac{\zeta(1-2r)}{1-2r}\,\frac{1}{s-(1-2r)} + \int_1^\infty H_M(z)z^{-s-1}\,dz. \tag{7}$$

From (3), (4) and (7) we get

$$\frac{\pi\zeta(s)}{\sin \pi s} = \sum_{k=2}^{\infty} \frac{(-1)^k\zeta(k)}{k(k-s)} + \frac{1-\gamma}{s-1} - \frac{1}{(s-1)^2} - \frac{1}{2s^2} - \frac{\log 2\pi}{2s} + \sum_{k=1}^{M} \frac{\zeta(1-2k)}{1-2k}\,\frac{1}{s-(1-2k)} + \int_1^\infty H_M(z)z^{-s-1}\,dz.$$

This formula which was shown to hold for s not an integer and for $1<\sigma<2$ now holds by analytic continuation for $\sigma>-M$ in view of the estimate on $H_M(z)$.

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