SOME ANALOGUES OF GLAUBERMAN'S Z^*-THEOREM

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Let x be an element of prime order, p, in a finite group, G, and let P be a p-Sylow subgroup of G containing x. Say that x satisfies the unique conjugacy condition (u.c.c.) relative to G and the p-Sylow subgroup P if x is not conjugate in G to any other element of P. Let $O_{2'}(G)$ denote the largest normal subgroup of G of odd order. Define the subgroup $Z^*(G)$ by the equation $Z^*(G)/O_{2'}(G) = Z(G/O_{2'}(G))$, the center of $G/O_{2'}(G)$. In the case that x is an involution satisfying the u.c.c., Glauberman [1] has shown that x lies in $Z^*(G)$. In particular, G is not nonabelian simple. Let $O_{p'}(G)$ denote the largest normal p'-subgroup of G. With an appropriate redefinition of $Z^*(G)$ by $Z^*(G)/O_{p'}(G) = Z(G/O_{p'}(G))$, Glauberman [1] has asked whether an element x of odd prime order satisfying the u.c.c. lies in $Z^*(G)$. However, a resolution of this question (if one exists) appears to be difficult. Nonetheless, it is possible to make a compromise. If we strengthen the hypothesis of the proposition we can obtain analogues of Glauberman's Z^*-theorem in which the conclusions are correspondingly stronger than $x \in Z^*(G)$. (These appear as Corollary 2 and Theorem 2 below.) The theorems may be of interest in that they represent nonsimplicity criteria involving only local hypotheses and have relatively elementary proofs. The author wishes to thank Professor Glauberman for his suggestion to modify Corollary 2 to the version appearing as Theorem 1.

1. The results.

THEOREM 1. *Let A be a solvable subgroup of G and suppose A lies in the center of every solvable subgroup of G which contains A. Then A lies in the center of G.*

COROLLARY 1. *Let A be a solvable subgroup of G and let $\mathcal{P}(A, G)$ be a property involving the embedding of A in G. Then if $\mathcal{P}(A, G)$ implies $\mathcal{P}(A, K)$ for every subgroup, K such that $A \leqq K \leqq G$ and if $\mathcal{P}(A, S)$ implies $A \leqq Z(S)$ for every solvable subgroup, S, of G containing A, then $A \leqq Z(G)$.*

COROLLARY 2. *Let x be an element of order p in the finite group, G, and suppose*

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(1) *x satisfies the u.c.c. relative to G and some p-Sylow subgroup P containing x.*

(2) *x centralizes every p'-group which it normalizes.*

Then x lies in the center of G.

COROLLARY 3. *Let G be a group whose p-Sylow subgroup P is a TI set. If $P\cap Z(N_G(P))$ is noncyclic then*

$$P \cap Z(N_G(P)) \leqq Z(G).$$

THEOREM 2. *Let x be an element of order p in the group G. Suppose*

(1) *x satisfies the u.c.c. relative to G and some p-Sylow subgroup P.*

(2) *If $X=\langle x\rangle$, then the collection of p'-groups normalized by X form a lattice.*

Then G has a normal p-complement and $x\in Z^*(G)$.

REMARKS. If x satisfies the u.c.c. and the identity is the only p'-subgroup normalized by x, both Corollary 2 and Theorem 2 apply. The condition (2) of Corollary 2 is encountered not infrequently. Suppose, for example, x is a p-element in a linear group over a field of characteristic p. If x acts with minimal polynomial of degree $p-1$ (or $p-2$ if p is a Fermat prime) then condition (2) holds for x.

It can be easily proved that the unique conjugacy condition for an element x of order p relative to G and some p-Sylow subgroup is equivalent to the requirement that x commute with none of its conjugates in G. Since the conjugates involved may result from conjugation by elements lying almost anywhere in G, in practice, this condition is as difficult to verify as the u.c.c. itself. Fortunately, this situation has been remedied by the following proposition recently proved by Glauberman, which provides a specific subgroup on which to test the u.c.c.

PROPOSITION. *Let P be a p-Sylow subgroup of G and let x be an element of order p lying in P. Define $J_0(P)$ to be the subgroup of P generated by the abelian subgroups of P having largest order. Suppose that either p is odd or that $p=2$ and* SL(2, 2) *is not involved in G. Then x satisfies the unique conjugacy condition relative to G and P if and only if*

$$x \in Z(N_G(J_0(P))).$$

Now suppose x satisfies the unique conjugacy condition relative to G and p-Sylow subgroup, P. Then $x\in P\leqq G$ and $x^g\in P$ implies $x^g=x$ for any $g\in G$. Suppose $x\in P^y$. Then if $x^u\in P^y$ for some $u\in G$, it follows that both $x^{y^{-1}}$ and $x^{uy^{-1}}$ lie in P and so $x=x^u$. Thus x also satisfies the unique conjugacy condition relative to G and any other

p-Sylow subgroup which contains x. Under these circumstances we say that x *satisfies the u.c.c. in* G.

It is easy to see that if x is a p-element satisfying the u.c.c. in G and $x \in H \leqq G$, then x satisfies the u.c.c. in H.

2. The proofs.

PROOF OF THEOREM 1. By hypothesis, A lies in the center of every solvable subgroup S containing A. Since A is itself solvable, the collection of such subgroups S is nonempty and so A is abelian. By induction, every proper subgroup K containing A has A in its center. Thus we may assume that $M = C_G(A)$ is the unique maximal subgroup of G containing A.

We now show that G has no normal solvable subgroups. Suppose $N \triangleleft G$ and N is solvable. Suppose $A \leqq N$. Then $A \leqq Z(N) \triangleleft G$. Suppose g is an element of G not centralizing A. Then $\langle g \rangle Z(N)$ is a solvable subgroup of G containing A which doesn't have A in its center. This contradicts our hypothesis and so every such g centralizes A, i.e., $A \leqq Z(G)$. On the other hand suppose $A \nleqq N$. Then if S/N is any solvable subgroup of G containing AN/N, S is a solvable subgroup of G containing A whence $A \leqq Z(S)$. Then $AN/N \leqq Z(S/N)$. Thus the hypotheses of the theorem inherit to AN/N as a subgroup of G/N. By induction $AN/N \leqq Z(G/N)$. In particular, AN is a solvable normal subgroup of G containing A and the arguments in the first part of this paragraph apply to yield $A \leqq Z(G)$. Thus we may assume G contains no nonidentity solvable normal subgroups.

Now suppose $M \triangleleft G$. But as $A \leqq Z(M)$, $Z(M)$ is a nontrivial solvable normal subgroup of G, contrary to the conclusion of the previous paragraph. Thus we must assume that M is self-normalizing.

Now suppose $1 \neq y \in M \cap M^g < M$. Then y centralizes both A and $g^{-1}Ag$ and so $\langle A, g^{-1}Ag \rangle \leqq C_G(y)$. If $C_G(y) = G$, $\langle y \rangle$ is a solvable normal subgroup of G, contrary to our previous remarks. Thus $C_G(y)$ lies in M. But also $g^{-1}Mg$ is unique maximal subgroup of G containing $g^{-1}Ag$. Since $g^{-1}Ag$ lies in $C_G(y) \leqq M$ we have $M = g^{-1}Mg$ contrary to our choice of M^g as intersecting M properly. Thus either $M = M^g$ or $M \cap M^g = 1$. By a famous theorem of Frobenius, M has a normal complement N in G. We now observe that A has order relatively prime to $|N|$. First let A_p be a nontrivial p-Sylow subgroup of A. Since A is abelian, $C_G(A) = M \leqq C_G(A_p)$. Clearly, $C_G(A_p) \cap N = 1$. Thus A_p acts in fixed point free manner on N and so $|N| \equiv 1 \bmod p$. This argument, repeated for each p-Sylow subgroup of A shows that N has order prime to A. Then since all complements of N in AN are conjugate, by the Schur-Zassenhaus theorem, an application of a

Frattini argument shows that A leaves invariant a nontrivial q-Sylow subgroup N_q of N. Then as AN_q is a solvable subgroup of G, by hypothesis $A \leqq Z(AN_q)$ contrary to $C_G(A) \cap N = 1$. This contradiction proves the theorem.

Corollary 1 follows immediately from Theorem 1 for the hypotheses imply the hypotheses of Theorem 1.

Proof of Corollary 2. Set $X = \langle x \rangle$, where x is an element of order p lying in a p-Sylow subgroup P, and x satisfies the u.c.c. relative to G and P. It then follows that x satisfies the u.c.c. in G.

Now suppose x centralizes every p'-group which it normalizes, so that x enjoys the hypotheses of Corollary 2. Suppose $X \leqq H \leqq G$. Let H_0 be a p'-subgroup of H normalized by x. Then H_0 *qua* subgroup of G, is centralized by x. Thus the pair (x, H) satisfies condition (2) of Corollary 2. From our previous remarks, x satisfies the u.c.c. in H as well as in G.

Thus the hypotheses of Corollary 2 which hold for the pair (X, G) inherits to pairs (X, H) where $X \leqq H \leqq G$. By Corollary 1, it suffices to prove this corollary when G is solvable.

Suppose $X \leqq N \lhd G$, where $N \neq G$. By induction on $|N|$, since our hypotheses hold for the pair (X, N), we have $X \leqq Z(N)$, and $X \leqq O_p(G) \leqq P$. Then G centralizes x since all conjugates of x lie in P under these circumstances. Thus we may assume X lies in no proper normal subgroup of G. Now $X \leqq C_G(O_{p'}(G))$ by hypothesis. As the latter is normal, it is G by our remarks above. But since $X \leqq Z(P)$, X also centralizes $O_p(G)$. As a consequence X centralizes the Fitting subgroup, $F(G)$, of G. Since the latter contains its own centralizer, $X \leqq F(G)$. Then $F(G) = G$ from our remark above. Since G is nilpotent, and $X \leqq Z(P)$, we have $X \leqq Z(G)$. This completes the proof of Corollary 2.

Proof of Corollary 3. By Theorem 1 it suffices to show that $A = P \cap Z(N_G(P))$ lies in the center of every solvable subgroup which contains it. Suppose $A \leqq S \leqq G$ where S is solvable. We wish to show that $A \leqq Z(S)$. Let P_1 be a p-Sylow subgroup of S containing A. Then, since P is a TI set, any p-Sylow subgroup of G containing P_1 coincides with P. Moreover, $N_G(P_1) \leqq N_G(P)$. Then $A \leqq Z(N_S(P_1)) \cap P_1$. Then if $S < G$, by induction on $|S|$, $Z(N_S(P_1)) \cap P_1$ lies in the center of S since it is noncyclic. Thus $A \leqq Z(S)$ as required. Thus we must assume that for some solvable subgroup S we have $G = S$, and if $A \leqq S_1 < S$, we have $A \leqq Z(S_1)$. Consequently, we may assume that $C_S(A)$ is the unique maximal subgroup of S containing A. Suppose $O_p(S) > 1$. Then as P is a TI set, $P \lhd S$. Then $A \leqq Z(N_S(P))$ implies $A \leqq Z(S)$. Thus we may suppose that $O_p(S) = 1$. Then $X = O_{p'}(S)$ is nontrivial since S is solvable. Now A is itself a TI set since $A \leqq Z(P)$. Since A is a noncyclic TI set normalizing X, it follows that A centralizes X. But as $O_p(S)=1$, $X \geqq C_S(X) \geqq A$. This is impossible since A is a p-group and X is a p'-group. This completes the proof of Corollary 3.

PROOF OF THEOREM 2. Set $X=\langle x\rangle$. If $X \leqq H \leqq G$, then the p'-subgroups of H which are normalized by X are closed under joins (group-theoretic unions) and also form a lattice. Also, from our remarks x satisfies the u.c.c. in H. Thus the hypotheses of the theorem on the pair (X, G) inherit to (X, H). If $H<G$, then H has a normal p-complement and $x \in Z^*(H)$ by induction. Thus if X is not normal in G, $N_G(X)$ is p-nilpotent. Let H^p denote the smallest normal subgroup of H such that H/H^p is a p-group, for any subgroup of G. By the theorem of Hall-Wielandt (Theorem 14.4.2 of [2]) since X is a weakly closed subgroup of the center of P, a p-Sylow subgroup of G, then $G/G^p \simeq N_G(X)/N_G(X)^p$. But from our previous remarks, $N_G(X) \geqq P$ and $N_G(X)$ is p-nilpotent, if X is not normal in G. Thus $N_G(X)/N_G(X)^p \simeq P$. Thus $G/G^p \simeq P$ so G has a normal p-complement G^p. Otherwise, we must assume $X \triangleleft G$. Then since x satisfies the u.c.c. and lies in $O_p(G)$, $X \leqq Z(G)$. Then X normalizes every q-Sylow subgroup Q of G so $Q \leqq M$ where M is the unique maximal p'-group normalized by X. Then M is a normal p'-Hall subgroup of G and so M is a normal p-complement in G. Thus whether or not X is normal in G, we obtain that G is p-nilpotent. The fact that $x \in Z^*(G)$ now follows from the fact that x is satisfying the u.c.c. in a p-nilpotent group. This completes the proof of Theorem 2.

REFERENCES

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