A SOLUTION TO THE NONVANISHING SEMI-CHARACTER EXTENSION PROBLEM

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1. **Introduction.** The problem referred to in the title is the following. If χ is a semicharacter defined on a subsemigroup S of a commutative semigroup T and if χ never takes on the value zero, when can χ be extended to a semicharacter of T which never takes on the value zero? The problem was considered in [1] and [2]. A sufficient condition for the extension of χ was given in [1], but there is an abundance of examples that show that this condition is not always necessary.

As in [1] and [2], we consider the condition:

$$(*)\quad (a, b, x) \in S \times S \times T \quad \text{and} \quad ax = bx \quad \text{imply} \quad \chi(a) = \chi(b).$$

The function α_χ defined in [1] is also used. In the present paper, however, we shall have an occasion to embed T in a larger semigroup U. Thus we adopt the more complete notation α_χ^T:

$$\alpha_\chi^T(x) = \begin{Bmatrix} 0 & \text{if } A_\chi^T(x) = \varnothing \\ \sup A_\chi^T(x) & \text{if } A_\chi^T(x) \neq \varnothing \end{Bmatrix},$$

where

$$A_\chi^T(x) = \{ | \chi(a)/\chi(b) |^{1/n} : bx^n yz = az \text{ with } (a, b, y, z) \in S \times S \times T \times T \}.$$

It is appropriate to consider another auxiliary function associated with χ. We define

$$\beta_\chi^T(x) = \inf\{ | \chi(a)/\chi(b) |^{1/n} : bx^n z = ayz \text{ with } (a, b, y, z) \in S \times S \times T \times T \}.$$

2. **Preliminary results.** An immediate consequence of the definition of the function α_χ^T is the following

LEMMA 1. *If χ is a nonvanishing semicharacter defined on a subsemigroup S of a commutative semigroup T, then the set*

$$I_\chi^T = \{ x \in T: \alpha_\chi^T(x) = 0 \}$$

is either empty or an ideal of T.

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Certain properties of the function β_χ^T which can be established by straightforward computations are listed in the following lemma.

LEMMA 2. *Suppose that χ is a nonvanishing semicharacter defined on a subsemigroup S of a commutative semigroup T. If $(c, x, y, z) \in S \times T \times T \times T$, then*
(i) $\beta_\chi^T(cx) = |\chi(c)| \beta_\chi^T(x)$,
(ii) $\beta_\chi^T(xy) \leqq \beta_\chi^T(x)\beta_\chi^T(y) \leqq \beta_\chi^T(x)$,
(iii) $\beta_\chi^T(x) = \beta_\chi^T(y)$ if $xz = yz$.
In particular, $0 \leqq \beta_\chi^T(x) \leqq 1$ *for each* $x \in T$.

LEMMA 3. *Suppose that χ is a positive semicharacter defined on a subsemigroup S of a commutative semigroup T and suppose that $\beta_\chi^T(x) > 0$ for each $x \in T$. Then $\beta_\chi^T(e) = 1$ if T has an identity e. Moreover, if T is without identity and if $U = T^e$ denotes T with an identity e adjoined, then $\beta_\chi^U(e) = 1$.*

PROOF. First, suppose that e is an identity for T. If $a \in S$, then $\beta_\chi^T(a) = \beta_\chi^T(ae) = \chi(a)\beta_\chi^T(e)$; hence $\beta_\chi^T(e) \neq 0$ since $\beta_\chi^T(a) > 0$. Part (ii) of Lemma 2 implies that $\beta_\chi^T(e) = 0$ or 1. Therefore, $\beta_\chi^T(e) = 1$.

Now suppose that T does not have an identity and let U denote T with an identity e adjoined. If $\beta_\chi^U(e) < 1$, then for some positive integer n we have that $be^n z = ayz$ where $(a, b, y, z) \in S \times S \times T \times T$ and $\chi(b) > \chi(a)$. However, this implies that $b^{m+1}zz^m = a^{m+1}(y^{m+1}z)z^m$ for any positive integer m and that $\beta_\chi^T(z) \leqq (\chi(a)/\chi(b))^{m+1}$. Since $\beta_\chi^T(z) > 0$, it follows that $\beta_\chi^U(e) = 1$.

LEMMA 4. *Suppose that χ is a positive semicharacter defined on a subsemigroup S of a commutative semigroup T with identity e. If $\beta_\chi^T(e) \neq 0$, then condition (*) is satisfied and $\alpha_\chi^T(x) \leqq 1$ for each x in T.*

PROOF. Since $\beta_\chi^T(e) \neq 0$, $\beta_\chi^T(e) = 1$. Suppose that $a \in S$. Using (i) of Lemma 2, we have that

$$\beta_\chi^T(a) = \beta_\chi^T(ae) = \chi(a)\beta_\chi^T(e) = \chi(a)$$

since χ is positive. Now suppose that $ax = bx$ where $(a, b, x) \in S \times S \times T$. Then $\chi(a) = \beta_\chi^T(a) = \beta_\chi^T(b) = \chi(b)$ and (*) is satisfied.

In order to prove that $\alpha_\chi^T(x) \leqq 1$ for each $x \in T$, suppose that $x \in T$ and suppose that $bx^n yz = az$ where $(a, b, y, z) \in S \times S \times T \times T$. Then

$$\chi(b) \geqq \chi(b)\beta_\chi^T(x^n y) = \beta_\chi^T(bx^n y) = \beta_\chi^T(a) = \chi(a),$$

which proves that $\alpha_\chi^T(x) \leqq 1$.

LEMMA 5. *Suppose that χ is a positive semicharacter defined on a* *subsemigroup S of a commutative semigroup T with identity e and suppose that $\beta_\chi^T(e)\neq 0$. Let θ be the natural map from T onto the maximal cancellative homomorphic image of T. Then $\psi\colon \theta(a)\to\chi(a)$ is a (well-defined) positive character of $\theta(S)$ and $\beta_\psi^{\theta(T)}(\theta(x))=\beta_\chi^T(x)$ for each $x\in T$.*

PROOF. Since θ is the natural map from T onto the maximal cancellative homomorphic image of T, $\theta(a)=\theta(b)$ if and only if $ax=bx$. Condition (*) holds by Lemma 4. Thus the mapping $\theta(a)\to\chi(a)$ is a well-defined mapping on $\theta(S)$. The verification of the equation $\beta_\psi^{\theta(T)}(\theta(x))=\beta_\chi^T(x)$ is also direct.

3. Principal results.

THEOREM. *Suppose that χ is a positive semicharacter defined on a subsemigroup S of a commutative semigroup T. Then χ can be extended to a positive semicharacter of T if and only if there is a subsemigroup P of $S\times T$ having the properties:*

(a) *if $x\in T$, there exists $a\in S$ such that $(a, x)\in P$;*
(b) *if $(a, x)\in P$, then $|\chi(a)|\leqq\beta_\chi^T(x)$.*

PROOF. Suppose that χ can be extended to a positive semicharacter ψ of T. Then

$$P = \{(a, x)\in S\times T\colon \chi(a)\leqq\psi(x)\}$$

is a subsemigroup having the properties (a) and (b) provided that we define $\psi(x)=1$ for all $x\in T$ if $\chi(x)=1$ for all $x\in S$. The main point here is that $\psi(x)\leqq\beta_\chi^T(x)$ for each $x\in T$.

Conversely, assume that P is a subsemigroup of $S\times T$ having the properties (a) and (b). Notice that these properties of P imply that $\beta_\chi^T(x)>0$ for each $x\in T$. If T does not have an identity element, let T^e denote T with an identity e adjoined. It follows from Lemma 3 that $P'=P\cup\{(a, e)\colon (a, x)\in P\}$ is a subsemigroup of $S\times T^e$ having properties (a) and (b) where T is replaced by T^e. Thus we may assume that T already has an identity e. By Lemma 5, we may assume that T is cancellative. Let G be a commutative group containing T. Since $\beta_\chi^T(e)\neq 0$, Lemma 4 implies that condition (*) is satisfied and that $\alpha_\chi^T(x)\leqq 1$ for each $x\in T$. If $I_\chi^T=\{x\in T\colon \alpha_\chi^T(x)=0\}$ is empty, then we know that χ can be extended to a positive semicharacter of T according to the theorem in [1]. If I_χ^T is not empty, we define $Q=\{(a, x)\in P\colon x\in I_\chi^T\}$. Now Q is a subsemigroup of P; indeed, Q is an ideal of P since I_χ^T is an ideal of T. Let π be the natural homomorphism from Q into G, that is, $\pi(a, x)=ax^{-1}$. Define $U=\{T, \pi(Q)\}$, the subsemigroup of G generated by T and $\pi(Q)$. Suppose that

$$be^n z = awz \quad \text{where} \quad (a, b, w, z) \in S \times S \times U \times U \quad \text{and} \tag{C}$$
$$n \text{ is a positive integer.}$$

Then $b = aw$ since U is cancellative and e is an identity element of U. Since $w \in U$, one of the following equations holds where $\chi(c) \leqq \beta_\chi^T(x)$ and $(c, x, y) \in S \times T \times T$: $w = y$, $w = cx^{-1}$, or $w = cx^{-1}y$. If $w = y$, then $\chi(b) \leqq \chi(a)$ since $\beta_\chi^T(e) = 1$. If $w = cx^{-1}$, then $bx = ac$, which implies that $x \notin I_\chi^T$; this leads to a contradiction since $ac = awx = bx$. Finally, if $w = cx^{-1}y$, then $bx = acy$ and $\chi(b)\beta_\chi^T(x) = \chi(a)\chi(c)\beta_\chi^T(y)$, so $\chi(b) \leqq \chi(a)\beta_\chi^T(y) \leqq \chi(a)$. We have shown that (C) implies that $\chi(b) \leqq \chi(a)$; hence $\beta_\chi^U(e) = 1$. It is easy to verify that I_χ^U is empty. Therefore, χ can be extended to a positive semicharacter of U and consequently to the subsemigroup T of U.

It is immediate that if the subsemigroup S of the commutative semigroup T is a homomorphic retract of T, then any positive semicharacter defined on S can be extended to a positive semicharacter of T. We now have a generalization of this.

Corollary 1. *Suppose that χ is a positive semicharacter defined on a subsemigroup S of a commutative semigroup T. If there is a homomorphism π from T into S such that $\chi\pi \leqq \beta_\chi^T$, then χ can be extended to a positive semicharacter of T.*

Proof. Define $P = \{(\pi(x), x) : x \in T\}$. Then P is a subsemigroup of $S \times T$ and P satisfies conditions (a) and (b) of Theorem 1.

Corollary 2. *Suppose that χ is a nonvanishing semicharacter defined on a subsemigroup S of a commutative semigroup T. Then χ can be extended to a nonvanishing semicharacter of T if and only if condition* (*) *holds and there is a subsemigroup P of $S \times T$ having properties* (a) *and* (b).

Proof. Observe that $\beta_\chi^T = \beta_{|\chi|}^T$ and apply Lemma 1 of [1].

4. **The insufficiency of a positive** β. Suppose that χ is a nonvanishing semicharacter defined on a subsemigroup S of a commutative semigroup T. The question whether the condition $\beta_\chi^T(x) > 0$ for each $x \in T$ is sufficient in order that χ can be extended to a nonvanishing semicharacter of T is natural to consider. A negative answer is given below. It is convenient now to use the additive notation. In fact, we exhibit a counterexample by using a semigroup T of the additive real numbers.

Let $L = \{a_{i,j}, x_k\}$ be a set of linearly independent real numbers where i and j range over the positive integers and k over the nonnegative integers. Let $S=\{a_{i,j}\}$ be the subsemigroup of the additive real numbers generated by the $a_{i,j}$'s and let $T=\{L,\ (ix_0+x_j-a_{i,j})\}$ be the subsemigroup of the additive reals generated by the set L and the numbers of the form $ix_0+x_j-a_{i,j}$ where i and j are positive integers.

Define a positive semicharacter χ on S by: $\chi(a_{i,j})=1/2^{ij}$. We show that $\beta_\chi^T(x)>0$ for each $x\in T$. First, suppose that $x=\sum_{k=0}^{r} n_k x_k$ where n_k is a nonnegative integer for each k, that is, suppose that $x\in X=\{x_0, x_1, x_2, \cdots\}$, the subsemigroup of T generated by the x_k's. Suppose that $a=\sum p_{i,j}a_{i,j}$ and $b=\sum q_{i,j}a_{i,j}$ where $1\leqq i,\ j\leqq m$ and $p_{i,j}$ and $q_{i,j}$ are nonnegative integers. If $b+nx=a+y$ where $y\in T$, then it follows from the definition of T and the linear independence of L that $p_{i,j}\leqq q_{i,j}$ if $j>r$ and that $p_{i,j}\leqq q_{i,j}+nn_0/i$ if $j\leqq r$. Thus $\chi(a)/\chi(b)\geqq(1/2)^{r(r+1)nn_0/2}$ and, therefore, $\beta_\chi^T(x)\geqq(1/2)^{r(r+1)n_0/2}$ since the latter number is independent of the choice of a and b in S. Now if t is an arbitrary element of T, there are elements c and d in S such that $c+t=x+d$ where $x\in X$. Hence

$$\beta_\chi^T(t)\geqq\beta_\chi^T(c+t)=\beta_\chi^T(x+d)=\chi(d)\beta_\chi^T(x)>0.$$

If ψ is an extension of χ to a positive semicharacter of T and if $\psi(x_k)=h_k$, then

$$\psi(ix_0+x_j-a_{i,j})=h_0^i h_j 2^{ij}=(2^j h_0)^i h_j>1$$

for appropriate i and j. Since χ cannot be extended to a positive semicharacter of T, χ cannot be extended to a nonvanishing semicharacter of T.

References

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