A NOTE ON HAAR-LIKE MEASURE FOR GROUP-EXTREMAL SEMIGROUPS

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Introduction. If S is a compact group-extremal affine semigroup, it is natural to ask how much of the group structure of the extreme points carries over to S. In [**4**], the author shows it is possible to extend sufficiently many unitary representations to separate points of S; in the abelian case, this leads to sufficiently many *affine* semicharacters to separate points of S. We investigate in this note the possibility of extending Haar measure to S. Using results in [**1**], [**2**], and [**3**], it can be seen that a compact affine semigroup which supports an idempotent measure must be of the form $X\times Y$ where X and Y are compact convex sets and multiplication is given by $(x, y)\cdot(x', y')=(x, y')$. Thus, one cannot hope in general to extend Haar measure and retain all of its properties. However, we will show that if S is a compact group-extremal affine semigroup, then there is a probability measure μ supported on S (i.e. μ-measure of each nonvoid open set is positive) satisfying

$$\int f(xy)\, d\mu(x) = \int f(yx)\, d\mu(x) = \int f(x)\, d\mu(x)$$

for each $y\in S$ and each *affine* continuous function on S.

Preliminaries. If S is a compact Hausdorff space, $C(S)$ will denote the Banach space of continuous complex-valued functions on S with the supremum norm, and $M(S)$ the Banach space of complex-valued regular Borel measures on S with variation norm. We use the notation $\mu(f)=\int f d\mu$ for $f\in C(S)$, $\mu\in M(S)$, and shall employ the Riesz-Kakutani Theorem often without comment. If $\mu\in M(S)$, $\mu\geqq 0$, $C(\mu)$ denotes the complement of the largest open set having μ-measure 0; we have $\mu(V)>0$ if and only if $V\cap C(\mu)\neq\emptyset$ holding for each open set V in S. $P(S)$ denotes the subset of $M(S)$ consisting of all nonnegative measures of norm 1.

LEMMA 1. *Let S and K be compact Hausdorff spaces and $f: S\rightarrow K$ a continuous function. For $\mu\in P(S)$, define $[f^*(\mu)](g)=\int g(f(x))d\mu(x)$ for $g\in C(K)$, then:*

(a) $f^*(\mu)\in P(K)$,

(b) $C(f^*(\mu))=f(C(\mu))$.

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PROOF. (a) is clear. Suppose $x_0 \notin f(C(\mu))$; there is a function $g \in C(K)$, $0 \leqq g \leqq 1$ where $g(y) > \frac{1}{2}$ on some open set V containing x_0, and $g(y) \equiv 0$ for $y \in f(C(\mu))$. We have

$$0 = \int_{C(\mu)} g(f(y))\, d\mu(y) = [f^*(\mu)](g) = \int g\, df^*(\mu)$$

$$\geqq \int_V g\, df^*(\mu) \geqq \tfrac{1}{2}[f^*(\mu)](V).$$

Thus, $[f^*(\mu)](V) = 0$ and $V \cap C(f^*(\mu)) = \varnothing$ so that $x_0 \notin C(f^*(\mu))$. Conversely, let $x_0 = f(y_0)$ where $y_0 \in C(\mu)$ and suppose $x_0 \notin C(f^*(\mu))$. There is a function $g \in C(K)$ $0 \leqq g \leqq 1$ and an open set V containing y_0 such that $g(f(y)) > \frac{1}{2}$ for $y \in V$ while $g(z) = 0$ for $z \in C(f^*(\mu))$. Thus $[f^*(\mu)](g) = 0$; however,

$$[f^*(\mu)](g) = \int g(f(y))\, d\mu(y) \geqq \int_V g(f(y))\, d\mu(y) \geqq \tfrac{1}{2}\mu(V) > 0$$

since $y_0 \in V \cap C(\mu)$. With this contradiction, the proof is complete.

LEMMA 2. *Let S be compact, Hausdorff and $\mu_i \in P(S)$ for $i = 1, 2, \cdots$. Then*

$$\lim_n \sum_{i=1}^{n} \frac{1}{2^i} \mu_i = \mu_0 \in P(S)$$

and

$$C(\mu_0) = \mathrm{Cl} \bigcup_{i=1}^{\infty} C(\mu_i).$$

(*The limit is in variation norm.*)

The proof consists of computations similar to those in Lemma 1, so we omit them. We point out, however, that Lemma 2 generalizes the well-known fact that a compact *separable* Hausdorff space supports a measure; this is done by taking μ_i to be a point mass concentrated at an element of a countable dense subset.

THEOREM 1. *Let S be a compact convex subset of a locally convex linear space, and $\mu \in P(S)$. Suppose that $C(\mu)$ contains the extreme points of S; then there is a measure $\nu \in P(S)$ satisfying* (a) *$C(\nu) = S$ and* (b) *$\int f d\nu = \int f d\mu$ for each continuous affine complex-valued function f on S.*

PROOF. Fix $n \geqq 1$ and define:

$$A_n = \left\{(\lambda_1, \cdots, \lambda_n) \in E^n : \lambda_i \geqq 0, \ \sum_{i=1}^{n} \lambda_i = 1\right\}. \tag{1}$$

Further, let $m_n \in P(A_n)$ such that $C(m_n) = A_n$. Let

$$K_n = A_n \times \mathop{\mathsf{P}}_{i=1}^{n} C_i \quad \text{where} \quad C_i = C(\mu) \quad \text{for } i = 1, 2, \cdots, n. \tag{2}$$

Finally, let $\nu_n \in P(K_n)$ be defined by

$$\nu_n(f) = \int\int \cdots \int f(z, x_1, \cdots, x_n)\, dm_n(z)\, d\mu(x_1) \cdots d\mu(x_n), \tag{3}$$

where $f \in C(K_n)$; then $C(\nu_n) = K_n$. The function $h_n : K_n \to S$ defined by

$$h_n\{(\lambda_1, \cdots, \lambda_n), x_1, \cdots, x_n\} = \sum_{i=1}^{n} \lambda_i x_i \tag{4}$$

is clearly continuous. We define $\sigma_n = h_n^*(\nu_n)$ where $h_n^*(\nu_n)$ is the measure in Lemma 1. Then we have,

$$\begin{aligned} C(\sigma_n) &= C(h_n^*(\nu_n)) = h_n(C(\nu_n)) = h_n(K_n) \\ &= \left\{y \in S : y = \sum_{i=1}^{n} \lambda_i x_i, \ \lambda_i \geqq 0, \ \sum_{i=1}^{n} \lambda_i = 1, \ x_i \in C(\mu)\right\} \end{aligned} \tag{5}$$

If f is a continuous affine function on S then

$$\begin{aligned} \sigma_n(f) &= [h_n^*(\nu_n)](f) = \int f(h_n(x)\, d\nu_n(x)) \\ &= \int\int \cdots \int f(h_n(z, x_1 \cdots x_n))\, dm_n(z)\, d\mu(x_1) \cdots d\mu(x_n) \\ &= \int\int \cdots \int f\left(\sum_{i=1}^{n} \lambda_i x_i\right) dm_n(\lambda_1 \cdots \lambda_n)\, d\mu(x_1) \cdots d\mu(x_n) \\ &= \int\int \cdots \int \sum_{i=1}^{n} \lambda_i f(x_i)\, dm_n(\lambda_1 \cdots \lambda_n)\, d\mu(x_1) \cdots d\mu(x_n) \\ &= \int f\, d\mu. \end{aligned}$$

Letting $\mu_0 = \lim_n \sum_{i=1}^{n} \sigma_i / 2^i$ we have, by Lemma 2, that

$$C(\mu_0) = \mathrm{Cl}\left(\bigcup_{n=1}^{\infty} C(\sigma_n)\right)$$

$$= \mathrm{Cl}\left(\bigcup_{n=1}^{\infty} \left\{y: y = \sum_{i=1}^{n} \lambda_i x_i, \lambda_i \geqq 0, \sum_{i=1}^{n} \lambda_i = 1, x_i \in C(\mu)\right\}\right) = S.$$

The last equality is a combination of (5), the fact that $C(\mu)$ contains the extreme points of S, and the Krein-Milman Theorem. Further, for each affine continuous function f,

$$\int f\, d\mu_0 = \lim_n \sum_{i=1}^{n} \frac{1}{2^i} \sigma_i(f) = \int f\, d\mu.$$

The measure μ_0 satisfies properties (a) and (b) of the theorem.

We apply Theorem 1 to the group-extremal affine semigroup S (for definitions see [3]) whose extreme points form the compact group G. Let m be Haar measure of the group G extended to the Borel sets of S by the formula $m(E) = m(E \cap G)$. Then $C(m) = G$ and $\int f dm = \int_G f(x) dm(x)$ for each continuous function f on S. By Theorem 1, we obtain a measure $\mu_0 \in P(S)$ where $C(\mu_0) = S$ and $\int f d\mu_0 = \int f dm$ for each affine continuous function f. If f is affine continuous on S, the function $f^y(x) \equiv f(xy)$ is another affine continuous function for each fixed $y \in S$. Thus, for $g \in G$ $\int f(xg) d\mu_0(x) = \int f^g d\mu_0 = \int f^g dm = \int f dm = \int f(x) d\mu_0(x)$. Since $y \rightarrow \int f(xy) d\mu_0(x)$ is also affine continuous, we have $\int f(xy) d\mu_0(x) = \int f(x) d\mu_0(x)$ for every $y \in S$. Similarly, $\int f(yx) d\mu_0(x) = \int f(x) d\mu_0(x)$ for each $y \in S$. We have proved

THEOREM 2. *If S is a compact group-extremal affine semigroup, there is a measure $\mu_0 \in P(S)$ for which $C(\mu_0) = S$ and $\int f(xy) d\mu_0(x) = \int f(yx) d\mu_0(x) = \int f(x) d\mu_0(x)$ for each $y \in S$ and each affine continuous $f \in C(S)$.*

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