EQUALITY OF MINIMAL AND MAXIMAL EXTENSIONS OF PARTIAL DIFFERENTIAL OPERATORS IN $L_p(R^n)$[1]

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It is known [1] that if Ω is a bounded domain, and $P=P(D)$ is a linear partial differential operator with constant coefficients, then every weak solution in $L_2(\Omega)$ with compact support in Ω, is also a strong solution.

In this paper, this result is generalized to show that the weak and strong solutions are equivalent for $\Omega=R^n$ and $L_p(R^n)$; $1\leqq p<\infty$, without the assumption that the solutions have compact support.

We consider a linear partial differential operator of order m, with constant coefficients: $P=P(D)=\sum_{|\alpha|\leqq m} a_\alpha D^\alpha$; $x\in R^n$. Here, $\alpha=(\alpha_1, \cdots, \alpha_n)$; the α_k are nonnegative integers and $|\alpha|=\sum\alpha_k$. $D=(D_1, \cdots, D_n)$; $D_k=(1/i)(\partial/\partial x_k)$, and $D^\alpha=D_1^{\alpha_1}\cdots D_n^{\alpha_n}$.

The double-barbed arrow "$\rightarrow$" will denote strong convergence in L_p, while the single-barbed arrow "$\rightharpoonup$" will denote weak convergence in L_p.

The convolution of two functions, u and v, will be denoted by $u*v=\int u(y)v(x-y)dy$.

Unless otherwise specified, all spaces will consist of functions on R^n, e.g., C_0^∞ means $C_0^\infty(R^n)$. Now, suppose $P=P(D)$ has domain $D(P)$, and range in L_p, $1\leqq p<\infty$. Let us take $D(P)=C_0^\infty$. It then follows that P is not closed, but satisfies the weaker property of being pre-closed.

Let P_0 be the closure (in L_p) of P on C_0^∞. P_0 is termed the minimal operator associated with P. The maximal operator P is defined as follows. $u\in L_p$ is in the maximal domain if $\exists v\in L_p$: $(u, P^*\phi)=(v, \phi)$, $\forall\phi\in C_0^\infty$. We then say $Pu=v$. We prove

THEOREM 1. *If $P=P(D)$ is a linear partial differential operator with constant coefficients on C_0^∞ into L_p, then the minimal and maximal extensions are equivalent for $1\leqq p<\infty$.*

For the proof, consider a function $\psi=\psi(x)$:

(i) $\psi\in C_0^\infty(K_2)$; where $K_2=\{x: |x|<2\}$.

(ii) $0\leqq\psi\leqq 1$, $(\forall x\in K_2)$; $\psi(x)\equiv 1$ for $|x|\leqq 1$. Then, form the sequence of functions $\psi^i=\psi^i(x)\equiv\psi(x/i)$.

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LEMMA 1. *For each* $\phi \in C_0^\infty$, *we have* $(\psi^i u)*\phi \to u*\phi$ *in* L_p, *for* $u \in L_p$; $1 \leqq p < \infty$.

PROOF. Since $u \in L_p$, for each $\epsilon > 0$, there exists an $M = M(\epsilon) > 0$:

$$\int_{|x|>M} |u|^p < \frac{\epsilon^p}{2}\cdot$$

Now, choose i large enough so that $i > M$. By Young's inequality,

$$\|[(1-\psi^i)u]*\phi\|_p \leqq \|\phi\|_1 \|(1-\psi^i)u\|_p.$$

But

$$\|(1-\psi^i)u\|_p^p = \int_{|x|\leq M} |(1-\psi^i)u|^p + \int_{|x|>M} |(1-\psi^i)u|^p$$
$$\leqq 2\int_{|x|>M} |u|^p < \epsilon^p.$$

Hence, we have proved the lemma.

PROOF OF THEOREM 1. Suppose for $u, v \in L_p$, that

$$(1) \qquad (u, P^*\phi) = (v, \phi); \qquad (\forall \phi \in C_0^\infty).$$

Let $j(x)$ denote a function:

$$j(x) \in C_0^\infty; \quad \int j = 1; \quad j \geqq 0; \quad \operatorname{supp} j = \{x: |x| \leqq 1\}.$$

Let $j_\nu = j_\nu(x) \equiv \nu^n j(\nu x)$, and form the mollifiers of u: $u_\nu = u_\nu(x) \equiv u*j_\nu$. Letting $f^{(x)}(y) \equiv f(x-y)$, we write $u_\nu = u*j_\nu = \int u(y) j_\nu(x-y)dy \equiv (u, j_\nu^{(x)})$. Now substitute $j_\nu^{(x)}$ for ϕ in (1) to obtain

$$(2) \qquad (u, P^*j_\nu^{(x)}) = (v, j_\nu^{(x)}) = v_\nu,$$

the mollifiers of v. But, if $u \in L_p$, then for any $\phi \in C_0^\infty$

$$(3) \quad P(u*\phi) \equiv P(D)(u*\phi) = (Pu)*\phi = (Pu, \phi^{(x)}) = (u, P^*\phi^{(x)}).$$

Therefore, for $u \in L_p$, (2) becomes

$$(2') \qquad Pu_\nu = P(u*j_\nu) = v_\nu; \qquad u_\nu, v_\nu \in C^\infty.$$

Now, apply Lemma 1 to $\phi = j_\nu$ and to $\phi = P^*j_\nu$ to obtain

$$(4) \qquad u_{i\nu} \equiv (\psi^i u)*j_\nu \to u*j_\nu = u_\nu$$

and

$$(5) \qquad Pu_{i\nu} = (\psi^i u)*P^*j_\nu \to u*P^*j_\nu = v_\nu.$$

But $(\psi^i u)$ has compact support, and, hence, $(\psi^i u)*j_\nu \equiv u_{i\nu} \in C_0^\infty$. Now, by (4), for each ν, there exists an integer $I_1 = I_1(\nu)$:

$$\|u_{i\nu} - u_\nu\|_p < 1/2^\nu; \qquad (\forall i \geqq I_1).$$

Furthermore, by (5), for each ν there exists an $I_2 = I_2(\nu)$:

$$\|Pu_{i\nu} - v_\nu\|_p < 1/2^\nu; \qquad (\forall i \geqq I_2).$$

Take $I \equiv I(\nu) \equiv \max[I_1, I_2]$; then, setting $S_\nu \equiv u_{I\nu}$, we have

(6) $$\|S_\nu - u_\nu\|_p < 1/2^\nu; \qquad \nu = 1, 2, \cdots,$$

and

(7) $$\|PS_\nu - v_\nu\|_p < 1/2^\nu; \qquad \nu = 1, 2, \cdots.$$

Therefore, we have exhibited a sequence $\{S_\nu\}$ such that

(i) $S_\nu \in C_0^\infty \equiv D(P)$,

(ii) $S_\nu \to u$; since $\|S_\nu - u\|_p \leqq \|S_\nu - u_\nu\|_p + \|u_\nu - u\|_p < 1/2^\nu + \|u_\nu - u\|_p \to 0$;

(iii) $PS_\nu \to v$; since $\|PS_\nu - v\|_p \leqq \|PS_\nu - v_\nu\|_p + \|v_\nu - v\|_p < 1/2^\nu + \|v_\nu - v\|_p \to 0$,

which proves that u is a strong solution of $Pu = v$.

It is interesting that if the spaces are restricted to functions on a bounded domain of R^n and a weak solution has compact support $\mathcal{S} \subset R^n$, then the proof that it is a strong solution in L_p $(1 \leqq p < \infty)$ is identical with the short proof of Hormander [1] for $p = 2$.

THEOREM 2. *Let $P = P(D)$ be a linear partial differential operator with constant coefficients on $C_0^\infty(\Omega) \to L_p(\Omega)$. Then, if u is a weak solution of $Pu = v$ such that* supp $u = \mathcal{S} \subset \Omega$, *$u$ is also a strong solution for $1 \leqq p < \infty$.*

PROOF. For $u_\nu = u * j_\nu$, we have $u_\nu \in C_0^\infty(\Omega)$; $\nu > 1/\lambda$, where λ is the distance from $\mathcal{S}$ to the complement of Ω. Then, $Pu_\nu = (u, P^* j_\nu^{(x)}) = (v, j_\nu^{(x)}) = v_\nu$. Hence, $u_\nu \in D(P)$, $u_\nu \to u$, and $Pu_\nu \to v$.

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