LINEAR MAPPINGS OF OPERATOR ALGEBRAS

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In (6) it was shown that a linear mapping ϕ of one C^*-algebra $\mathfrak{A}$ with identity into another which carries unitary operators into unitary operators is a C^*-homomorphism followed by multiplication by the unitary operator $\phi(I)$, i.e. $\phi(A)=\phi(I)\rho(A)$, $\phi(A^*)=\phi(A)^*$, and $\rho(A^2)=\rho(A)^2$ for each A in $\mathfrak{A}$. We continue in that spirit here, with the unitary group replaced first by an arbitrary semigroup contained in the unit sphere, then by the semigroup of regular contractions. By a C^*-algebra we shall mean a uniformly closed self-adjoint algebra of bounded linear operators on some Hilbert space, which contains the identity operator.

LEMMA 1. *Let $\mathfrak{B}$ be a normed algebra containing a multiplicative semigroup $\mathfrak{S}$ with the following properties*: (i) *the linear span of $\mathfrak{S}$ is $\mathfrak{B}$*; (ii) $\sup\{\|s\|:s\in\mathfrak{S}\}=K<\infty$. *For x in $\mathfrak{B}$, define $\|x\|_{\mathfrak{S}}$ to be* $\inf\{\sum_1^n |a_j| : x=\sum_1^n a_j s_j,\ s_j\in\mathfrak{S},\ a_j \text{ complex},\ n\geqq 1\}$. *Then $\|\cdot\|_{\mathfrak{S}}$ is a normed algebra norm on $\mathfrak{B}$ such that $\|\cdot\|\leqq K\|\cdot\|_{\mathfrak{S}}$. Furthermore, if $\mathfrak{S}$ and $\mathfrak{J}$ are multiplicative semigroups in the normed algebras $\mathfrak{B}$ and $\mathfrak{C}$ resp., each satisfying* (i) *and* (ii), *and if ϕ is a linear mapping of $\mathfrak{B}$ into $\mathfrak{C}$ such that $\phi(\mathfrak{S})\subseteq\mathfrak{J}$, then for each x in $\mathfrak{B}$, $\|\phi(x)\|_{\mathfrak{J}}\leqq\|x\|_{\mathfrak{S}}$.*

PROOF. Verify.

Let $\mathfrak{A}$ be a C^*- algebra and let $\mathfrak{S}$ be a multiplicative semigroup contained in the unit sphere of $\mathfrak{A}$. Suppose that the linear span of $\mathfrak{S}$ is $\mathfrak{A}$ and that $\|A\|_{\mathfrak{S}}=\|A\|$ whenever A is a regular element of $\mathfrak{A}$. For example $\mathfrak{S}$ could be the group of unitary operators, the semigroup of regular contractions, or the entire unit sphere of $\mathfrak{A}$.

LEMMA 2. *Let ϕ be a linear mapping of $\mathfrak{A}$ into a C^*-algebra $\mathfrak{B}$ such that $\phi(I)=I$ and ϕ maps $\mathfrak{S}$ into the unit sphere of $\mathfrak{B}$. Then ϕ is a self-adjoint mapping, i.e. $\phi(A^*)=\phi(A)^*$.*

PROOF. We argue as in [2, Lemma 8]. Let A be a self-adjoint element of $\mathfrak{A}$ of norm 1. Then $\phi(A)=B+iC$, where B and C are self-adjoint elements of $\mathfrak{B}$. If $C\neq 0$, let b be a positive number in the spectrum of C (otherwise consider $-C$). Choose a positive integer n such that $(1+n^2)^{1/2}<b+n$. Then since $A+inI$ is regular, $\|A+inI\|=(1+n^2)^{1/2}<b+n\leqq\|iC+inI\|\leqq\|B+i(C+nI)\|=\|\phi(A+inI)\|$

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$\leqq \|A+inI\|_{\mathcal{S}} = \|A+inI\|$, a contradiction. It follows that ϕ is a self-adjoint mapping.

THEOREM 1. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be C^*-algebras and let $\mathcal{S}$(resp. $\mathfrak{J}$) be a multiplicative semigroup contained in the unit sphere of $\mathfrak{A}$ (resp. $\mathfrak{B}$). Suppose that $\|A\|_{\mathcal{S}} = \|A\|$ (resp. $\|B\|_{\mathfrak{J}} = \|B\|$) whenever A (resp. B) is a regular element of $\mathfrak{A}$ (resp. $\mathfrak{B}$). Let ϕ be a one-to-one linear mapping of $\mathfrak{A}$ onto $\mathfrak{B}$ such that $\phi(I) = I$, ϕ maps $\mathcal{S}$ into the unit sphere of $\mathfrak{B}$, and ϕ^{-1} maps $\mathfrak{J}$ into the unit sphere of $\mathfrak{A}$. Then ϕ is a C^*-isomorphism.*

PROOF. By Lemma 2, ϕ is a self-adjoint mapping. If A is a self-adjoint element of $\mathfrak{A}$ then $A+iI$ is regular and $(\|\phi(A)\|^2+1)^{1/2} = \|\phi(A+iI)\| \leqq \|A+iI\|_{\mathcal{S}} = \|A+iI\| = (\|A\|^2+1)^{1/2}$, so that $\|\phi(A)\| \leqq \|A\|$. Similarly $\|\phi^{-1}(B)\| \leqq \|B\|$ for each self-adjoint element B of $\mathfrak{B}$. Thus ϕ is an isometry of the Jordan algebra of self-adjoint elements of $\mathfrak{A}$ onto the Jordan algebra of self-adjoint elements of $\mathfrak{B}$ [**3**]. By a theorem of Kadison [**3**, Theorem 2], ϕ is a C^*-isomorphism.

The theorem shows that isometries of C^*-algebras which preserve the identity are C^*-isomorphisms [**2**, Theorem 7].

We now consider the semigroup $R_1(\mathfrak{A})$ of all regular contractions of a C^*-algebra $\mathfrak{A}$, i.e. the set of all invertible elements of $\mathfrak{A}$ of norm at most one. Let ϕ be a linear mapping of $\mathfrak{A}$ into a C^*-algebra $\mathfrak{B}$ such that $\phi(I) = I$ and $\phi(R_1(\mathfrak{A})) \subseteq R_1(\mathfrak{B})$. By Lemma 2, ϕ is a self-adjoint mapping and clearly $\phi(R(\mathfrak{A})) \subseteq R(\mathfrak{B})$, where $R(\mathfrak{A})$ denotes the group of all regular elements of the C^*-algebra $\mathfrak{A}$.

In case $\mathfrak{A} = \mathfrak{B}$ is a matrix algebra, it is known [**5**, Theorem 2.1] that the weaker hypothesis $\phi(R(\mathfrak{A})) \subseteq R(\mathfrak{B})$ implies that ϕ is a Jordan homomorphism (i.e. preserves squares) followed by multiplication by a fixed regular element. We next show that this result does not generalize to arbitrary C^*-algebras except in a very special case, namely for commutative $\mathfrak{B}$.

EXAMPLE. Let $\mathfrak{A}$ be any C^*-algebra and let $\mathfrak{B} = M_2(\mathfrak{A})$ be the C^*-algebra of all 2 by 2 matrices with entries in $\mathfrak{A}$. Let ζ be any automorphism of $\mathfrak{A}$. Define a mapping ϕ of $\mathfrak{A}$ into $M_2(\mathfrak{A})$ by the formula

$$\phi(A) = \begin{pmatrix} A & A - \zeta(A) \\ 0 & A \end{pmatrix}, \quad (A \in \mathfrak{A}).$$

Then clearly ϕ is a linear mapping such that $\phi(I) = I$, but it is easy to check that $\phi(R(\mathfrak{A})) \subseteq R(M_2(\mathfrak{A}))$ and that ϕ is not a Jordan homomorphism unless ζ is the identity automorphism.

PROPOSITION. *Let ϕ be a linear mapping of a C^*-algebra $\mathfrak{A}$ into a commutative C^*-algebra $\mathfrak{B}$ such that $\phi(R(\mathfrak{A})) \subseteq R(\mathfrak{B})$. Then there is a*

C^*-homomorphism ρ of $\mathfrak{A}$ into $\mathfrak{B}$ and an element B in $R(\mathfrak{B})$ such that $\phi(A)=B\rho(A)$ for each A in $\mathfrak{A}$.

PROOF. Set $\rho(A)=\phi(I)^{-1}\phi(A)$. Then $\rho(I)=I$ and $\rho(R(\mathfrak{A}))\subseteq R(\mathfrak{B})$ and it suffices to show that ρ is a C^*-homomorphism. Since $\rho(I)=I$, the condition $\rho(R(\mathfrak{A}))\subseteq R(\mathfrak{B})$ is equivalent to $\mathrm{Sp}(\rho(A))\subseteq \mathrm{Sp}(A)$ for each A in $\mathfrak{A}$, where $\mathrm{Sp}(A)$ denotes the spectrum of the operator A. If U is any unitary operator in $\mathfrak{A}$ then $\mathrm{Sp}(\rho(U))$ is a subset of the unit circle. Since $\mathfrak{B}$ is commutative, $\rho(U)$ is normal, hence unitary. The result follows from [**6**, Corollary 2].

We now return to the semigroup of regular contractions. By the remarks following Theorem 1 we may assume our mappings are self-adjoint.

LEMMA 3. *Let ϕ be a linear self-adjoint mapping of a C^*-algebra $\mathfrak{A}$ into a C^*-algebra $\mathfrak{B}$ such that $\phi(R(\mathfrak{A}))\subseteq R(\mathfrak{B})$ and $\phi(I)=I$. Then* (i) *if P is a projection in $\mathfrak{A}$, then $\phi(P)$ is a projection in $\mathfrak{B}$*; (ii) *if P and Q are orthogonal projections in $\mathfrak{A}$, then $\phi(P)$ and $\phi(Q)$ are orthogonal projections in $\mathfrak{B}$.*

PROOF. (i) if P is a projection, then $\phi(P)$ is a self-adjoint operator with spectrum contained in the two point set $\{0, 1\}$. (ii) if U is a self-adjoint unitary operator in $\mathfrak{A}$ then $\phi(U)$ is self-adjoint and unitary in $\mathfrak{B}$. An operator T is a projection if and only if $I-2T$ is self-adjoint and unitary. Let P and Q be orthogonal projections in $\mathfrak{A}$ and set $U=I-2P$, $V=I-2Q$. The orthogonality of P and Q implies that U and V commute. Hence UV is also a self-adjoint unitary operator. Thus $\phi(UV)=I-2(\phi(P)+\phi(Q))$ is self-adjoint and unitary so that $\phi(P)+\phi(Q)$ is a projection. It follows that $\phi(P)\phi(Q)=0$.

LEMMA 4. *Let ϕ be a linear self-adjoint mapping of a commutative C^*-algebra $\mathfrak{A}$ into a C^*-algebra $\mathfrak{B}$ such that $\phi(R(\mathfrak{A}))\subseteq R(\mathfrak{B})$ and $\phi(I)=I$. Then $\|\phi\|=1$.*

PROOF. Let A be a positive element of $\mathfrak{A}$. Then $\phi(A)$ is self-adjoint and since $\mathrm{Sp}(\phi(A))\subseteq \mathrm{Sp}(A)$ it follows that $\phi(A)$ is positive. Thus ϕ is a positive mapping. By results of Stinespring [**7**, Theorems 1 and 4], there is a Hilbert space K, a $*$-representation ρ of $\mathfrak{A}$ on K and an isometry V of H into K ($\mathfrak{B}$ acts on H) such that $\phi(A)=V^*\rho(A)V$ for all A in $\mathfrak{A}$. Thus if $A\in\mathfrak{A}$, then $\|\phi(A)\|=\|V^*\rho(A)V\|\leqq\|A\|$.

Recall that a von Neumann algebra is a C^*-algebra which is closed in the weak operator topology [**1**, p. 33].

THEOREM 2. *Let ϕ be a linear mapping of a von Neumann algebra M*

into a C^*-algebra $\mathcal{B}$ such that $\phi(R_1(M))\subseteq R_1(\mathcal{B})$ and $\phi(I)=I$. Then ϕ is a C^*-homomorphism.

PROOF. As noted above, ϕ is self-adjoint and $\phi(R(M))\subseteq R(\mathcal{B})$. Let A be a self-adjoint element of M of norm 1. The von Neumann algebra M_0 generated by A is commutative and if $\epsilon>0$ there exist orthogonal projections $P_1, P_2, \cdots, P_n$ in M_0 and real numbers $r_1, r_2, \cdots, r_n$ such that $\|A-\sum_1^n r_iP_i\|<\epsilon$ [1, p. 3]. By several applications of the preceding two lemmas and after a computation one obtains $\|\phi(A)^2-\phi(A^2)\|<2\in(2+\epsilon)$. Since ϵ was arbitrary $\phi(A)^2=\phi(A^2)$ for each self-adjoint A in M of norm 1. It follows trivially that ϕ is a C^*-homomorphism.

We note that the theorem holds with an identical proof in case M is an AW^*-algebra [4].

REMARKS. 1. It is an open question as to whether Theorem 2 is true when M is a C^*-algebra. Since the conclusion, i.e. $\phi(A)^2=\phi(A^2)$, need only hold for self-adjoint operators A, there is no loss of generality in assuming M to be commutative. Then by Lemma 4 we have $\|\phi\|=1$.

2. The author believes that a solution to the following special case would shed considerable light on the problem: let $\mathcal{A}$ be a commutative C^*-algebra acting on a Hilbert space H, and let P be a projection operator on H, say mapping H onto a subspace K. Let ϕ be the mapping $\phi(A)=PA$ of $\mathcal{A}$ into the bounded operators on K. The reason for this belief is the relation of the mapping $A\rightarrow PA$ to the results of Stinespring quoted above, and to normal dilations of operators.

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