GOLAB'S THEOREM

CURTIS M. FULTON

J. Witkowski in [3] proved a theorem of S. Golab which gives a characterization of the sphere in E^3. In this paper a simpler proof of Golab's theorem is presented. The more direct approach involved should make the geometry simpler to visualize. First we state necessary

DEFINITIONS. (I) A curve on a surface of class C^1 is called B-straight if the tangent planes to the surface along the curve remain always perpendicular to a fixed direction. (II) A curve on a surface of class C^1 is called B-plane if the tangent planes to the surface along the curve are all parallel to a fixed direction.

We wish to prove the following

THEOREM. *If every geodesic of a regular surface of class C^3 is B-plane but not B-straight, then the surface is part of a sphere.*

PROOF (INDIRECT). According to the hypothesis of our theorem each geodesic is B-plane, that is there exists for each geodesic a constant unit vector $\boldsymbol{V}$ which is perpendicular to the surface normal $\boldsymbol{N}$ along the geodesic. Differentiation of $\boldsymbol{V}\cdot\boldsymbol{N}=0$ with respect to the arc length yields $\boldsymbol{V}\cdot d_s\boldsymbol{N}=0$ where $d_s\boldsymbol{N}$ is not identically zero since by hypothesis the geodesic is not B-straight. We assume that $d_s\boldsymbol{N}\neq 0$ at the points under discussion and the later development will show that other points need not be considered. Now for a geodesic, the principal normal $\boldsymbol{n}$ is equal to the surface normal $\boldsymbol{N}$. Thus the unit tangent $\boldsymbol{t}$ satisfies the relation $\boldsymbol{V}\cdot\boldsymbol{t}=\cos\theta=\text{const}$ which can be shown by differentiation. This means that the geodesics are helices, some fixed angle θ belonging to each geodesic. Also, at a point of a geodesic the vectors $\boldsymbol{V}$, $\boldsymbol{t}$, and $d_s\boldsymbol{N}$ are in the tangent plane and we find for the curvature κ, which along a geodesic is the same as the normal curvature,

$$(1)\qquad -\kappa = \boldsymbol{t}\cdot d_s\boldsymbol{N} = \pm\ |d_s\boldsymbol{N}|\ \sin\theta.$$

Using the third fundamental form [2, p. 103] along a geodesic, we have

$$(2)\qquad d_s\boldsymbol{N}\cdot d_s\boldsymbol{N} = -\kappa_1\kappa_2 + (\kappa_1+\kappa_2)\kappa,$$

which allows us to change (1) to

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$$\kappa^2 = \sin^2\theta[-\kappa_1\kappa_2 + (\kappa_1 + \kappa_2)\kappa]. \tag{3}$$

Assume the existence of a neighborhood R_1 on the surface that does not contain any umbilic points and therefore is covered by lines of curvature. Within R_1 there must be a neighborhood R_2 where the curvature K does not vanish. If such an R_2 would not exist, the set of points in R_1 with $K=0$ would be dense and a continuity argument would show that $K=0$ at all points of R_1. In this case, however, we have geodesics that are B-straight [**3**, Lemma 1]. Such geodesics being ruled out by our hypothesis we now take a point P in R_2. Consider the geodesic through P in the principal direction corresponding to κ_1. Its curvature at P is also κ_1 which we know to be different from zero. But then (2) shows that $d_s\boldsymbol{N}\neq 0$ and consequently the arguments leading to (3) are valid. We can infer from (3) that $\sin^2\theta=1$. Then along this geodesic $\sin^2\theta$ will continue to equal 1 and continuity shows that the value of κ found in (3) will be equal to κ_1 throughout. Hence the geodesic coincides with the line of curvature in R_2. Also, in R_2 the lines of curvature can be used as coordinate curves [**1**, p. 56]. Since they are geodesics, R_2 has curvature $K=0$ [**1**, p. 45].

It follows that K is identically zero in R_1. Therefore, a neighborhood R_1 without umbilics satisfying the hypothesis of our theorem cannot exist. Rather, the set of umbilics on the surface is dense and we are dealing with part of a sphere [**3**, Lemma 2].

References

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University of California, Davis