ON FUNCTION SPACES OF STRATIFIABLE SPACES AND COMPACT SPACES

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1. **Introduction.** Throughout, for any topological spaces X and Y, Y^X will denote the space of continuous functions from X to Y with the compact-open topology,[2] unless otherwise stated. Since stratifiable[3] spaces, which we studied in [2], have many of the desirable properties of metrizable spaces (every CW-complex of Whitehead is stratifiable—see Theorem 7.2 in [2] or Theorem 8.1 in [3]; furthermore, it is easily seen that metrizable spaces are stratifiable and every stratifiable space is paracompact and perfectly normal), we naturally *questioned whether Y^X is stratifiable given that X is compact Hausdorff and Y is stratifiable. We will now answer this question negatively.* However, the stratifiable space Y of our example is not a CW-complex and thus the following question still remains unanswered: Is K^X stratifiable whenever X is compact Hausdorff and K is a CW-complex? Whenever K is a countable CW-complex and X is compact metrizable we will however show that K^X is a cosmic[4] space (hence K^X is hereditarily Lindelöf, thus paracompact, and hereditarily separable) whenever K^X has the pointwise topology or the compact-open topology.

We will also give a *negative answer to the following question of Stone* [9]: Is Y^I a normal space whenever Y is compact Hausdorff and finite-dimensional (in the covering sense)?

Throughout we use the terminology of Kelley [6], except that all our topological spaces are T_1.

2. **Theorems and proofs.** Throughout this section, let I denote the closed unit interval.

THEOREM 1. *There exists a stratifiable space X such that X^I is not a normal space. Furthermore X is a 2-dimensional* (*in any sense*—ind, Ind, dim) *cosmic space.*[4]

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[2] For K a compact subset of X and V an open subset of Y, $W(K, V)=\{f\in Y^X \mid f(K)\subset V\}$. A *subbase for the compact-open topology* in Y^X is the family of all $W(K, V)$, with K compact and V open ($K\subset X$, $V\subset Y$).

[3] A topological space X is a *stratifiable* space if X is T_1 and, to each open $U\subset X$, one can assign a sequence $\{U_n\}_{n=1}^{\infty}$ of open subsets of X such that, for all n, $U_n^-\subset U$, $U_n\subset V_n$ whenever $U\subset V$, and $\bigcup_{n=1}^{\infty} U_n=U$. This definition is equivalent to Definition 1.3 in [3] (thus our stratifiable spaces are the same as the M_3-spaces of Ceder [3]).

[4] A topological space X is cosmic if it is the continuous image of a separable metrizable space.

PROOF. Let $X=\{(x, y)|x \text{ and } y \text{ are real numbers and } y\geqq 0\}$. For each $(x, 0)\in X$ and angle α, $0<\alpha<\pi$, let $L(x, \alpha)$ be the ray through $(x, 0)$ lying in X whose angle of inclination with the positive x-axis is α. For each $(x, y)\in X$, with $y\neq 0$, and real number $r>0$, let $D((x, y), r)$ denote the circular disc centered at (x, y) and with radius r, and let $C((x, y), r)$ denote the boundary of $D((x, y), r)$.

Let X have the topology τ with the following neighborhood system:

(a) If $(x, y)\in X$ and $y\neq 0$, a neighborhood of (x, y) is any $D((x, y), r)$ with $0<r<y$.

(b) A neighborhood of $(w, 0)\in X$ is a set of the form $N(w, \sigma, \partial) = S(w, \pi-\sigma, \partial)\cup\{(w, 0)\}\cup S(w, \sigma, \partial)$, where

$$S(w, \alpha, \partial) = \{(x, y) \in X \mid |x - w| < \partial, y < (x - w) \tan \alpha\}$$

(where $\partial>0$, $0<\alpha<\pi$, and $\alpha\neq\pi/2$). Note that $S(w, \alpha, \partial)$ is a region bounded by a right triangle, lying to the right of $(w, 0)$ if $0<\alpha<\pi/2$, to the left of $(w, 0)$ if $\pi/2<\alpha<\pi$.

It is easily seen that (X, τ) is stratifiable:[5] For each open $U\subset X$ and positive integer n, let $U_n=U_n'\cup U_n''$, where

$$U_n' = \bigcup \{D((x, y), 1/2n) \mid D((x, y), 1/n) \subset U\},$$
$$U_n'' = \bigcup \{N(x, 1/2n, 1/2n) \mid N(x, 1/n, 1/n) \subset U\}.$$

A simple argument shows that $(U_n'')^-\subset U$. Obviously, $(U_n')^-\subset U$, $\bigcup_{n=1}^{\infty} U_n=U$ and $U_n\subset V_n$ whenever $U\subset V$. Consequently (X, τ) is stratifiable (see footnote 1).

It is clear that $\operatorname{ind} X=2$, and it can be seen that $\dim X=2=\operatorname{Ind} X$. Letting $Y=\{(x, y)\in X|y>0\}$ and $R=X-Y$ then both Y and R are separable metrizable subspaces of X and thus X is the one-to-one continuous image of the topological sum[6] of Y and R (See Example 12.1 in [8]). Hence X is a cosmic space.

We will now show that X^I is not a normal space: Let F be the set of all functions $f_x\in X^I$ $(x\in X)$ such that f_x maps I in a "natural fashion" onto the arc C_x of the circle $C((x, 1), 1)$ with unit length, and such that $f_x(\frac{1}{2})=(x, 0)$ (simply lay the unit interval I around $C((x, 1), 1)$ so that the center of I coincides with the point $(x, 0)$). Then

(a) X^I is not hereditarily separable (this was first observed by Professor E. A. Michael): F is a discrete subspace of X^I since

[5] Actually one can show that X is M_1 (see Definition 1.1 in [3]) by the method of proof used in Example 9.2 of [3].

[6] A topological space M is the topological sum of the family $\{X_\alpha\}_{\alpha\in L}$ of topological spaces if $M=\bigcup_{\alpha\in L}X_\alpha\times\{\alpha\}$ with $X_\alpha\{\alpha\}$ homeomorphic to X_α for each $\alpha\in L$.

$$\{f_x\} = W(I, N(x, \pi/6, 1)) \cap W(\{0\}, S(x, 5\pi/6, 1)) \cap W(\{1\}, S(x, \pi/6, 1))$$

for each $x \in X$. Since F is discrete and uncountable, X^I is not hereditarily separable.

(b) F is a closed subset of X^I: By 2.4, 2.5 and 4.71 in [1], a net $\{f_x\}_{x\in A}$ converges to $h \in X^I$ if and only if $\{f_x\}_{x\in A}$ converges continuously[7] to h. Hence, if $\{f_x\}_{x\in A}$ converges continuously to $h\in X^I$ then $\{f_x(t)\}_{x\in A}$ converges to $h(t)$ for each $t\in I$, and thus $h\in F$. (Let $a = h(\frac{1}{2})$. By the definition of the functions $f_x \in F$ it is clearly seen that for each $t \in I$, $\{f_x(t) \mid x \text{ is a real number}\}$ is a subset of a horizontal straight line. Thus one can easily see that $h = f_a \in F$: Certainly, for each $t\in I$, $h(t)$ and $f_a(t)$ are in the same horizontal line since $h(t)$ is the limit point of the net $\{f_x(t)\}_{x\in A}$; thus it is obvious that $h(t) = f_a(t)$.)

(c) There exists a separable subspace Z of X^I which contains F: For each pair (r, t) of rational numbers with $r<t$ and $t-r<1$, let f_{rt} be the function in X^I such that f_{rt} maps the closed interval $I_r = [0, (1+r-t)/2]$ in a "natural fashion" onto the arc C_r of $C((r, 1), 1)$ with length $(1+r-t)/2$ and $f_{rt}((1+r-t)/2) = (r, 0)$, f_{rt} similarly maps $I_t = [(1+t-r)/2, 1]$ onto the arc C_t of $C((t, 1), 1)$ with length $(1+r-t)/2$ and $f_{rt}((1+t-r)/2) = (t, 0)$, and f_{rt} maps $I-(I_r \cup U_t)$ onto $\{(x, 0) \mid r<x<t\}$ in a "natural fashion."

Now let $Z = F \cup D$, where $D = \{f_{rt} \in X^I \mid r \text{ and } t \text{ are rational numbers}, r<t \text{ and } t-r<1\}$. We show that D is a dense subset of Z: Given any function $f_x \in F$, let $\{r(n)\}_{n=1}^{\infty}$ be a decreasing sequence of rational numbers converging to x and let $\{q(n)\}_{n=1}^{\infty}$ be an increasing sequence of rational numbers converging to x, such that $r(1)-q(1)<1$. Then the sequence $\{f_{q(n)r(n)}\}_{n=1}^{\infty}$ of functions in D continuously converges to f_x: Let $\{t_\nu\}_{\nu\in C}$ be a net of points in I which converges to $t\in I$. If $t \neq \frac{1}{2}$ then $f_x(t) \notin \{(x, 0) \mid x \text{ is a real number}\}$ and obviously the net $\{g_n(t_\nu)\}$ converges to f_x, where $g_n = f_{q(n)r(n)}$ for each n, since the subspace $\{(x, y) \in X \mid y>0\}$ of X is also a subspace of the cartesian plane with the usual topology. If $t = \frac{1}{2}$, then $f_x(t) = (x, 0)$ and one easily sees that $\{g_n(t_\nu)\}$ converges to $f_x(t)$ from the definition of the functions g_n and the neighborhood system of $(x, 0)$. Since D is countable then Z is separable.

(d) X^I is not a normal space: Let E be the closure of Z in X^I. Then E is separable and contains a subset F of cardinality $2^{\aleph_0}$ without a limit point in E (since F is a discrete closed subspace of X^I). By Theorem 1 in [5], E is not normal.

[7] The net $\{f_x\}_{x\in A}$ converges continuously to h if the net $\{f_x(y_\nu)\}$ converges to $f(y)$ whenever the net $\{y_\nu\}_{\nu\in C}$ converges to y (we omit the domain of composite nets).

Consequently, X^I is not normal (E is a closed subspace of X^I which is not normal).

THEOREM 2. *Let K be a countable CW-complex and X a compact metrizable space. Then K^X is a cosmic space (see footnote 4) whenever K^X has the pointwise topology (or the compact-open topology[8]).*

PROOF. Let M be the topological sum of the countable family $\{C_n\}_{n=1}^{\infty}$ of finite subcomplexes of K (i.e. $M=\bigcup_{n=1}^{\infty} C_r \times \{n\}$). Then $M^X = \bigcup_{n=1}^{\infty} C_n^X$, since X is compact, and hence M^X with the pointwise topology, or the compact-open topology, is separable metrizable, because of Theorem 1 in [7].

Now we define a map $f: M \to K$ by $f(k, n) = k$ for each $(k, n) \in M$. Clearly f is continuous (indeed f is a quotient map). Then we define a map $\phi: M^X \to K^X$ by $\phi(h) = f \circ h$. Clearly ϕ is an onto map. Furthermore $\phi | C_n^X$ (the restriction of ϕ to C_n^X) is a homeomorphism and M^X is the disjoint topological union of $\{C_n^X\}_{n=1}^{\infty}$. Hence ϕ is continuous whenever both M^X and K^X have the pointwise topology (or the compact-open topology).

THEOREM 3. *There exists a 2-dimensional (in the covering sense) compact Hausdorff space Y of weight[9] $2^{\aleph_0}$ such that Y^I is not a normal space.*

PROOF. Let X, F and Z be the spaces constructed in the proof of Theorem 1, and let Y be the Wallman compactification of X (since X is a normal space, Y is also the Stone-Cěch compactification of X—see Exercise T, p. 169, in [6] for pertinent definitions and results). Then Y has the same covering dimension of X ($\dim X = 2$) and weight$^{\aleph_0}$ (since the topology of X is carried onto a base for the topology of Y, the cardinality of the topology of X is $2^{\aleph_0}$, and any base for the topology of X must contain some $N(w, \sigma, \partial)$ for each real number w). Then we get that X^I and hence Z are subspaces of Y^I.

We show that Z is an F_σ-subset of Y^I: For each positive integer n, let $F_n = \{f_w \in F \mid -n \leqq w \leqq n\}$. Then F_n is a closed subset of Y^I—we essentially repeat part (b) of the proof of Theorem 1 (keeping in mind that Y^I is Hausdorff (so nets converge to at most one point) and $\{(x, y) \mid -n \leqq x \leqq n\}$ is a compact subspace of Y for each $y \geqq 0$). Then Z is the union of countably many closed subsets of Y^I, since $Z - F$ is

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[8] Actually, if K^X has the compact-open topology then a stronger version of Theorem 2 is already known—see Definitions 1.1 and 1.2, and results (I) and (J) of [8].

[9] The weight of a topological space X is the minimum of the cardinal numbers of the open bases for the topology on X.

countable, and hence Z is an F_σ-subset of Y^I. Consequently Y^I is not a normal space, since F_σ-subsets of normal spaces are normal and Z (an F_σ-subset of Y^I) is not a normal space.

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