ON FUNCTION SPACES OF STRATIFIABLE SPACES AND COMPACT SPACES

CARLOS J. R. BORGES

1. Introduction. Throughout, for any topological spaces X and Y, \( Y^X \) will denote the space of continuous functions from X to Y with the compact-open topology, unless otherwise stated. Since stratifiable spaces, which we studied in [2], have many of the desirable properties of metrizable spaces (every CW-complex of Whitehead is stratifiable—see Theorem 7.2 in [2] or Theorem 8.1 in [3]; furthermore, it is easily seen that metrizable spaces are stratifiable and every stratifiable space is paracompact and perfectly normal), we naturally questioned whether \( Y^X \) is stratifiable given that X is compact Hausdorff and Y is stratifiable. We will now answer this question negatively. However, the stratifiable space Y of our example is not a CW-complex and thus the following question still remains unanswered: Is \( K^X \) stratifiable whenever X is compact Hausdorff and K is a CW-complex? Whenever K is a countable CW-complex and X is compact metrizable we will however show that \( K^X \) is a cosmic space (hence \( K^X \) is hereditarily Lindelöf, thus paracompact, and hereditarily separable) whenever \( K^X \) has the pointwise topology or the compact-open topology.

We will also give a negative answer to the following question of Stone [9]: Is \( Y^1 \) a normal space whenever Y is compact Hausdorff and finite-dimensional (in the covering sense)?

Throughout we use the terminology of Kelley [6], except that all our topological spaces are T_1.

2. Theorems and proofs. Throughout this section, let I denote the closed unit interval.

**Theorem 1.** There exists a stratifiable space X such that \( X^I \) is not a normal space. Furthermore X is a 2-dimensional (in any sense—ind, Ind, dim) cosmic space.

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2 For K a compact subset of X and V an open subset of Y, \( W(K, V) = \{f \in Y^X \mid f(K) \subseteq V\} \). A subbase for the compact-open topology in \( Y^X \) is the family of all \( W(K, V) \), with K compact and V open (\( K \subseteq X, V \subseteq Y \)).

3 A topological space X is a stratifiable space if X is T_1 and, to each open \( U \subseteq X \), one can assign a sequence \( \{U_n\}_{n=1}^\infty \) of open subsets of X such that, for all \( n \), \( U_1 \subseteq U, U_n \subseteq V_n \) whenever \( U \subseteq V \), and \( \bigcup_{n=1}^\infty U_n = U \). This definition is equivalent to Definition 1.3 in [3] (thus our stratifiable spaces are the same as the \( M^X \)-spaces of Ceder [3]).

4 A topological space X is cosmic if it is the continuous image of a separable metrizable space.
Proof. Let \( X = \{(x, y) \mid x \text{ and } y \text{ are real numbers and } y \geq 0\} \). For each \((x, 0) \in X\) and angle \( \alpha, 0 < \alpha < \pi \), let \( L(x, \alpha) \) be the ray through \((x, 0)\) lying in \( X\) whose angle of inclination with the positive \( x\)-axis is \( \alpha \). For each \((x, y) \in X\), with \( y \neq 0 \), and real number \( r > 0 \), let \( D((x, y), r) \) denote the circular disc centered at \((x, y)\) and with radius \( r \), and let \( C((x, y), r) \) denote the boundary of \( D((x, y), r) \).

Let \( X\) have the topology \( \tau \) with the following neighborhood system:

(a) If \((x, y) \in X\) and \( y \neq 0\), a neighborhood of \((x, y)\) is any \( D((x, y), r) \) with \( 0 < r < y\).

(b) A neighborhood of \((w, 0) \in X\) is a set of the form \( N(w, \sigma, \partial) = S(w, \pi - \sigma, \partial) \cup \{(w, 0)\} \cup S(w, \sigma, \partial) \), where

\[
S(w, \alpha, \partial) = \{ (x, y) \in X \mid |x - w| < \partial, y < (x - w) \tan \alpha \}
\]

(where \( \partial > 0, 0 < \alpha < \pi \), and \( \alpha \neq \pi / 2 \)). Note that \( S(w, \alpha, \partial) \) is a region bounded by a right triangle, lying to the right of \((w, 0)\) if \( 0 < \alpha < \pi / 2 \), to the left of \((w, 0)\) if \( \pi / 2 < \alpha < \pi \).

It is easily seen that \((X, \tau)\) is stratifiable. For each open \( U \subset X\) and positive integer \( n\), let \( U_n = U_n' \cup U_n'' \), where

\[
U_n' = \bigcup \{ D((x, y), 1/2n) \mid D((x, y), 1/n) \subset U \},
\]

\[
U_n'' = \bigcup \{ N(x, 1/2n, 1/2n) \mid N(x, 1/n, 1/n) \subset U \}.
\]

A simple argument shows that \((U_n')^{-} \subset U\). Obviously, \((U_n')^{-} \subset U\), \( U_{n=1}^\infty U_n = U \) and \( U_n \subset V_n \) whenever \( U \subset V \). Consequently \((X, \tau)\) is stratifiable (see footnote 1).

It is clear that \( \text{ind } X = 2\), and it can be seen that \( \dim X = 2 = \text{Ind } X \).

Letting \( Y = \{(x, y) \in X \mid y > 0\} \) and \( R = X - Y \) then both \( Y \) and \( R \) are separable metrizable subspaces of \( X \) and thus \( X \) is the one-to-one continuous image of the topological sum \(^6\) of \( Y \) and \( R \) (See Example 12.1 in [8]). Hence \( X \) is a cosmic space.

We will now show that \( X^I \) is not a normal space: Let \( F \) be the set of all functions \( f_x \in X^I \) (\( x \in X \)) such that \( f_x \) maps \( I \) in a “natural fashion” onto the arc \( C_x \) of the circle \( C((x, 1), 1) \) with unit length, and such that \( f_x(1/2) = (x, 0) \) (simply lay the unit interval \( I \) around \( C((x, 1), 1) \) so that the center of \( I \) coincides with the point \((x, 0)\)). Then

(a) \( X^I \) is not hereditarily separable (this was first observed by Professor E. A. Michael): \( F \) is a discrete subspace of \( X^I \) since

\[^5\] Actually one can show that \( X \) is \( M_1 \) (see Definition 1.1 in [3]) by the method of proof used in Example 9.2 of [3].

\[^6\] A topological space \( M \) is the topological sum of the family \( \{X_\alpha\}_{\alpha \in L} \) of topological spaces if \( M = \bigcup_{\alpha \in L} X_\alpha \times \{\alpha\} \) with \( X_\alpha \{\alpha\} \) homeomorphic to \( X_\alpha \) for each \( \alpha \in L \).
\[ \{f_x\} = W(I, N(x, \pi/6, 1)) \cap W(\{0\}, S(x, S\pi/6, 1)) \cap W(\{1\}, S(x, \pi/6, 1)) \]
for each \(x \in X\). Since \(F\) is discrete and uncountable, \(X^I\) is not hereditarily separable.

(b) \(F\) is a closed subset of \(X^I\): By 2.4, 2.5 and 4.71 in [1], a net \(\{f_x\}_{x \in A}\) converges to \(h \in X^I\) if and only if \(\{f_x\}_{x \in A}\) converges continuously to \(h\). Hence, if \(\{f_x\}_{x \in A}\) converges continuously to \(h \in X^I\) then \(\{f_x(t)\}_{x \in A}\) converges to \(h(t)\) for each \(t \in I\), and thus \(h \in F\). (Let \(a = h(\frac{1}{2})\). By the definition of the functions \(f_x \in F\) it is clearly seen that for each \(t \in I\), \(\{f_x(t) | x \text{ is a real number}\}\) is a subset of a horizontal straight line. Thus one can easily see that \(h = f_a \in F\): Certainly, for each \(t \in I\), \(h(t)\) and \(f_a(t)\) are in the same horizontal line since \(h(t)\) is the limit point of the net \(\{f_x(t)\}_{x \in A}\); thus it is obvious that \(h(t) = f_a(t)\).)

(c) There exists a separable subspace \(Z\) of \(X^I\) which contains \(F\): For each pair \((r, t)\) of rational numbers with \(r < t\) and \(t - r < 1\), let \(f_{rt}\) be the function in \(X^I\) such that \(f_{rt}\) maps the closed interval \(I_r = [0, (1+r-t)/2]\) in a “natural fashion” onto the arc \(C_{r}\) of \(C((r, 1), 1)\) with length \((1+r-t)/2\) and \(f_{rt}((1+r-t)/2) = (r, 0)\). \(f_{rt}\) similarly maps \(I_t = [(1+r-t)/2, 1]\) onto the arc \(C_{t}\) of \(C((t, 1), 1)\) with length \((1+r-t)/2\) and \(f_{rt}((1+r-t)/2) = (t, 0)\), and \(f_{rt}\) maps \(I - (I_r \cup U_t)\) onto \(\{(x, 0) | r < x < t\}\) in a “natural fashion.”

Now let \(Z = F \cup D\), where \(D = \{f_{rt} \in X^I | r \text{ and } t \text{ are rational numbers, } r < t \text{ and } t - r < 1\}\). We show that \(D\) is a dense subset of \(Z\): Given any function \(f_x \in F\), let \(\{r(n)\}_{n=1}^\infty\) be a decreasing sequence of rational numbers converging to \(x\) and let \(\{q(n)\}_{n=1}^\infty\) be an increasing sequence of rational numbers converging to \(x\), such that \(r(1) - q(1) < 1\). Then the sequence \(\{f_{q(n)r(n)}\}_{n=1}^\infty\) of functions in \(D\) continuously converges to \(f_x\): Let \(\{t_r\}_{r \in C}\) be a net of points in \(I\) which converges to \(t \in I\). If \(t \neq \frac{1}{2}\) then \(f_x(t) \notin \{(x, 0) | x \text{ is a real number}\}\) and obviously the net \(\{g_n(t_r)\}\) converges to \(f_x\), where \(g_n = f_{q(n)r(n)}\) for each \(n\), since the subspace \(\{(x, y) \in X | y > 0\}\) of \(X\) is also a subspace of the cartesian plane with the usual topology. If \(t = \frac{1}{2}\), then \(f_x(t) = (x, 0)\) and one easily sees that \(\{g_n(t_r)\}\) converges to \(f_x(t)\) from the definition of the functions \(g_n\) and the neighborhood system of \((x, 0)\). Since \(D\) is countable then \(Z\) is separable.

(d) \(X^I\) is not a normal space: Let \(E\) be the closure of \(Z\) in \(X^I\). Then \(E\) is separable and contains a subset \(F\) of cardinality \(2^{\aleph_0}\) without a limit point in \(E\) (since \(F\) is a discrete closed subspace of \(X^I\)). By Theorem 1 in [5], \(E\) is not normal.

\footnote{The net \(\{f_x\}_{x \in A}\) converges continuously to \(h\) if the net \(\{f_x(y)\}\) converges to \(f(y)\) whenever the net \(\{y_t\}_{t \in C}\) converges to \(y\) (we omit the domain of composite nets).}
Consequently, \( X^I \) is not normal (\( E \) is a closed subspace of \( X^I \) which is not normal).

**Theorem 2.** Let \( K \) be a countable CW-complex and \( X \) a compact metrizable space. Then \( K^X \) is a cosmic space (see footnote 4) whenever \( K^X \) has the pointwise topology (or the compact-open topology\(^8\)).

**Proof.** Let \( M \) be the topological sum of the countable family \( \{C_n\}_{n=1}^\infty \) of finite subcomplexes of \( K \) (i.e. \( M = \bigcup_{n=1}^\infty C_n \times \{n\} \)). Then \( M^X = \bigcup_{n=1}^\infty C_n^X \), since \( X \) is compact, and hence \( M^X \) with the pointwise topology, or the compact-open topology, is separable metrizable, because of Theorem 1 in \[7].

Now we define a map \( f: M \to K \) by \( f(k, n) = k \) for each \( (k, n) \in M \). Clearly \( f \) is continuous (indeed \( f \) is a quotient map). Then we define a map \( \phi: M^X \to K^X \) by \( \phi(h) = f \circ h \). Clearly \( \phi \) is an onto map. Furthermore \( \phi|C_n^X \) (the restriction of \( \phi \) to \( C_n^X \)) is a homeomorphism and \( M^X \) is the disjoint topological union of \( \{C_n^X\}_{n=1}^\infty \). Hence \( \phi \) is continuous whenever both \( M^X \) and \( K^X \) have the pointwise topology (or the compact-open topology).

**Theorem 3.** There exists a 2-dimensional (in the covering sense) compact Hausdorff space \( Y \) of weight\(^9\) \( 2^{\aleph_0} \) such that \( Y^I \) is not a normal space.

**Proof.** Let \( X, F \) and \( Z \) be the spaces constructed in the proof of Theorem 1, and let \( Y \) be the Wallman compactification of \( X \) (since \( X \) is a normal space, \( Y \) is also the Stone-C\v{e}ch compactification of \( X \)—see Exercise T, p. 169, in \[6\] for pertinent definitions and results). Then \( Y \) has the same covering dimension of \( X \) (\( \dim X = 2 \)) and weight\(^9\) \( 2^{\aleph_0} \) (since the topology of \( X \) is carried onto a base for the topology of \( Y \), the cardinality of the topology of \( X \) is \( 2^{\aleph_0} \), and any base for the topology of \( X \) must contain some \( N(w, \sigma, \partial) \) for each real number \( w \)). Then we get that \( X^I \) and hence \( Z \) are subspaces of \( Y^I \).

We show that \( Z \) is an \( F_\sigma \)-subset of \( Y^I \): For each positive integer \( n \), let \( F_n = \{f_w \subseteq F | -n \leq w \leq n \} \). Then \( F_n \) is a closed subset of \( Y^I \)—we essentially repeat part (b) of the proof of Theorem 1 (keeping in mind that \( Y^I \) is Hausdorff (so nets converge to at most one point) and \( \{(x, y) | -n \leq x \leq n \} \) is a compact subspace of \( Y \) for each \( y \geq 0 \)). Then \( Z \) is the union of countably many closed subsets of \( Y^I \), since \( Z - F \) is

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\(^8\) Actually, if \( K^X \) has the compact-open topology then a stronger version of Theorem 2 is already known—see Definitions 1.1 and 1.2, and results (I) and (J) of \[8\].

\(^9\) The weight of a topological space \( X \) is the minimum of the cardinal numbers of the open bases for the topology on \( X \).
countable, and hence $Z$ is an $F_\sigma$-subset of $Y^I$. Consequently $Y^I$ is not a normal space, since $F_\sigma$-subsets of normal spaces are normal and $Z$ (an $F_\sigma$-subset of $Y^I$) is not a normal space.

**Bibliography**

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*University of California, Davis*