ON STABLE VECTOR BUNDLES OVER REAL PROJECTIVE SPACES

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If X is a connected, finite CW-complex, we can define $(KO)^{\sim}(X)$ to be $[X, BO]$ (base-point preserving homotopy classes of maps). Recall [2] that if $x \in (KO)^{\sim}(X)$, the geometrical dimension of x (abbreviated g.dim x) can be defined to be the smallest nonnegative integer k such that a representative of x factors through $BO(k)$. If ξ is a vector bundle over X, the class in $(KO)^{\sim}(X)$ of a classifying map for ξ is called the stable class of ξ.

Let P^n denote real projective n-space, and let $x_n \in (KO)^{\sim}(P^n)$ denote the stable class of the canonical line bundle γ_n over P^n.

If m is a positive integer, let $\phi(m)$ denote the number of integers k such that $0 < k \leqq m$ and $k \equiv 0, 1, 2, 4 \pmod 8$. The purpose of this note is to prove the following theorem:

THEOREM. *Let $n-2<2m<2n$. Then for any integers r, s,*

$$\text{g.dim}\,(2^{\phi(m)} r s x_n) \leqq \max\{m+1, \text{g.dim}\,(2^{\phi(m)} r x_n)\}.$$

Applications are given below. We first need two lemmas.

LEMMA 1. *Let $X \supset Y$ be a pair of finite CW-complexes, and let $p: X \to X/Y$ be the natural projection. Let $x \in (KO)^{\sim}(X)$ be in the image of p^*. Then there exists $y \in (KO)^{\sim}(X/Y)$ such that $p^* y = x$ and*

$$\text{g}\cdot\text{dim}\, y \leqq \max\{1 + \dim Y, \text{g}\cdot\text{dim}\, x\}.$$

PROOF. Let $m = \max\{1 + \dim Y, \text{g.dim}\, x\}$. Then there exists an m-plane bundle ξ over X whose stable class is x. From the exact sequence

$$(KO)^{\sim}(Y) \overset{i^*}{\leftarrow} (KO)^{\sim}(X) \overset{p^*}{\leftarrow} (KO)^{\sim}(X/Y)$$

where i^* is induced by the inclusion $i: Y \subset X$, it follows that $i^* x = 0$, and so $i^*\xi$ is stably trivial. But since $m > \dim Y$, $i^*\xi$ must be trivial. Hence, from the exact sequence of sets

$$[Y, BO(m)] \overset{i^*}{\leftarrow} [X, BO(m)] \overset{p^*}{\leftarrow} [X/Y, BO(m)]$$

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there exists an m-plane bundle η over X/Y such that $p^*\eta=\xi$. Take y to be the stable class of η.

LEMMA 2. *Let X be the suspension of a finite CW-complex (or, more generally, a co-H-space with homotopy co-unit and co-inverse). Then for each $n\geqq 0$, $\{x\in(KO)^\sim(X)\,|\,\text{g.dim}\,x\leqq n\}$ is a subgroup of $(KO)^\sim(X)$.*

PROOF. Let x and y be the stable classes of n-plane bundles ξ and η, respectively, over X. Then $x+y$ is the stable class of $\xi\oplus\eta$. Let $f,\ g:X\to BO(n)$ be classifying maps for ξ and η, respectively. The following diagram is homotopy commutative:

$$\begin{array}{ccccccc} X & \xrightarrow{\Delta} & X\times X & \xrightarrow{f\times g} & BO(n)\times BO(n) & \to & BO(2n) \\ & {\scriptstyle\mu}\searrow & \uparrow\cup & & \uparrow\cup & & \uparrow \\ & & X\vee X & \xrightarrow{f\vee g} & BO(n)\vee BO(n) & \to & BO(n) \end{array}$$

Here Δ is the diagonal map, μ is the pinch map, $BO(n)\vee BO(n)\to BO(n)$ is the folding map, and $BO(n)\times BO(n)\to BO(2n)$, $BO(n)\to BO(2n)$ are induced by the inclusions $O(n)\times O(n)\subset O(2n)$, $O(n)\subset O(2n)$, respectively. The composition of the maps of the top row is a classifying map for $\xi\oplus\eta$. Since it factors through $BO(n)$, $\xi\oplus\eta$ is equivalent to the Whitney sum of an n-plane bundle with a trivial bundle. Hence g.dim $(x+y)\leqq n$.

Since $[X,\ BO(n)]$ is a group under track multiplication, given $f:X\to BO(n)$, there exists $g:X\to BO(n)$ such that the composition

$$X\to X\vee X\xrightarrow{f\vee g} BO(n)\vee BO(n)\to BO(n)$$

is homotopically trivial. Hence given an n-plane bundle ξ over X, the above diagram implies that there exists an n-plane bundle η over X with $\xi\oplus\eta$ trivial. Hence g.dim $(-x)\leqq n$.

PROOF OF THEOREM. P^n/P^m is the Thom space of $(m+1)\gamma_{n-m-1}$ over P^{n-m-1}. Since $m+1>n-m-1$, this bundle admits a nowhere zero cross-section, and so P^n/P^m is a suspension.

If $i:P^m\subset P^n$ is the standard inclusion, $i^*x_n=x_m$. By [1], $2^{\phi(m)}rx_m=0$, and so $2^{\phi(m)}rx_n$ is in the image of p^*, where $p:P^n\to P^n/P^m$ denotes the natural projection. By Lemma 1, there exists $y\in(KO)^\sim(P^n/P^m)$ such that $p^*y=2^{\phi(m)}rx_n$, and g.dim $y\leqq\max\{m+1,\ \text{g.dim}\,(2^{\phi(m)}rx_n)\}$. By Lemma 2, g.dim $sy\leqq$ g.dim y for any integer s. Hence

$$\text{g.dim}\ p^*sy\leqq\max\ \{m+1,\ \text{g.dim}\ (2^{\phi(m)}rx_n)\}.$$

APPLICATIONS. (i) g.dim $(16s+7)x_{13}=7$ for all integers s.

(ii) g.dim $(16s+8)x_{15}=8$ for all integers s.

PROOF. It follows from the fact that P^{15} admits 8 everywhere linearly independent vector fields [1] that g.dim $16x_{15} \leqq 7$. Hence g.dim $16x_{13} \leqq 7$. Applying the Theorem with $n=13$, $m=6$, $r=2$, we obtain g.dim $(16s\, x_{13}) \leqq 7$. By [3, Lemma 2.2], g.dim $(qx_r) \leqq k$ if and only if g.dim $(-q+k)x_r \leqq k$. Hence, g.dim $(-16s+7)x_{13} \leqq 7$. The reverse inequality follows by a Stiefel-Whitney class argument. Since s is arbitrary, replacing s by $-s$ yields (i).

(ii) is obtained similarly, applying the Theorem with $n=15$, $m=7$, and $r=1$.

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