SHORTER NOTES

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A NOTE ON SARD'S THEOREM IN BANACH SPACES

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Suppose $E$ and $F$ are real Banach spaces, $f: E \rightarrow F$ is a $C^1$-map, and $Df(x)$ denotes the Fréchet derivative of $f$ at $x$. A point $x \in E$ is called a critical point if the linear map $Df(x): E \rightarrow F$ is not surjective. In [2] Smale gave the following generalization of Sard's Theorem:

**Theorem.** Suppose $f: E \rightarrow F$ is a $C^n$-map $(n \geq 1)$, and $Df(x)$ is a Fredholm operator for each $x \in E$. Then

$$\dim(\ker Df(x)) - \dim(\coker Df(x)) = \text{index}(f)$$

does not depend on $x$, and if $n > \text{index}(f)$, then the image under $f$ of the critical points is a set of first category.

Kupka [1] showed that the Fredholm assumption on the derivatives is essential. His example of a real valued function is fairly involved. The purpose of this note is to point out that a very simple example of the same phenomenon can be given if we do not insist that the function be real valued.

**Example.** Let $E$ denote all real valued continuous functions on $[0, 1]$, and suppose that $f: E \rightarrow E$ is defined by $f(x) = x^3$. Then $f$ is of class $C^{\infty}$, in fact $Df(x)h = 3x^2 h$, and $D^n f = 0$ when $n \geq 4$. From the formula for $Df(x)$ it follows that $x$ is a critical point iff $x$ vanishes at some point. Moreover a critical point is an interior point of the set of critical points iff it takes on both positive and negative values. Clearly $f$ maps the set of critical points onto itself, and the interior points onto the interior points. Hence the image of the critical points contains an open set.

**Bibliography**


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1218