SHORTER NOTES

The purpose of this department is to publish very short papers of an unusually elegant and polished character, for which there is no other outlet.

A NOTE ON SARD'S THEOREM IN BANACH SPACES

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Suppose E and F are real Banach spaces, $f\colon E\to F$ is a C^1-map, and $Df(x)$ denotes the Fréchet derivative of f at x. A point $x\in E$ is called a critical point if the linear map $Df(x)\colon E\to F$ is not surjective. In [2] Smale gave the following generalization of Sard's Theorem:

THEOREM. *Suppose $f\colon E\to F$ is a C^n-map $(n\geq 1)$, and $Df(x)$ is a Fredholm operator for each $x\in E$. Then*

$$\dim(\ker Df(x)) - \dim(\operatorname{coker} Df(x)) = \operatorname{index}(f)$$

does not depend on x, and if $n>\operatorname{index}(f)$, then the image under f of the critical points is a set of first category.

Kupka [1] showed that the Fredholm assumption on the derivatives is essential. His example of a real valued function is fairly involved. The purpose of this note is to point out that a very simple example of the same phenomenon can be given if we do not insist that the function be real valued.

EXAMPLE. Let E denote all real valued continuous functions on $[0, 1]$, and suppose that $f\colon E\to E$ is defined by $f(x)=x^3$. Then f is of class C^∞, in fact $Df(x)h=3x^2h$, and $D^nf=0$ when $n\geq 4$. From the formula for $Df(x)$ it follows that x is a critical point iff x vanishes at some point. Moreover a critical point is an interior point of the set of critical points iff it takes on both positive and negative values. Clearly f maps the set of critical points onto itself, and the interior points onto the interior points. Hence the image of the critical points contains an open set.

BIBLIOGRAPHY

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2. S. Smale, *An infinite dimensional version of Sard's theorem*, Amer. J. Math. **87** (1965), 861–866.

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