ON THE ASYMPTOTIC DENSITY OF THE k-FREE INTEGERS

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A positive integer is said to be k-free ($k \geqq 2$) if it contains no perfect kth power factor greater than 1. Let $S(x)$ be the number of k-free integers $\leqq x$. It is well known that

$$\delta_k = \lim_{x \to \infty} \frac{S(x)}{x} = \frac{1}{\zeta(k)},$$

where $\zeta(k)$ is the Riemann zeta function. In fact, Evelyn and Linfoot [2] have shown that

$$\begin{aligned} T(x) &= S(x) - \frac{[x]}{\zeta(k)} \\ &= O(x^{1/k} \exp[-b(\log x \log\log x)^{1/2}]), \qquad b = ak^{-3/2} \end{aligned} \tag{1}$$

and a is an absolute constant >0. They have also shown that

$$T(x) \neq o(x^{1/2k}). \tag{2}$$

The elementary result that $T(x) = O(x^{1/k})$ goes back to at least 1885 [3, p. 47].

Recently, some interest has been shown in the Schnirelmann density of the k-free integers:

$$D_k = \inf_{n>0} \frac{S(n)}{n}.$$

Duncan [5] has shown that

$$D_2 \leqq \delta_2 < D_3 \leqq \delta_3 < \cdots < D_k \leqq \delta_k < D_{k+1} \leqq \cdots.$$

Since $S(n)/n$ is initially greater than δ_k, one may reasonably ask if $D_k = \delta_k$. Rogers [6] has shown that for $k=2$ this is not so, and in fact

$$D_2 = \frac{53}{88} < \frac{6}{\pi^2} = \delta_2.$$

The method used in [6] is computational and sheds no light on the possible equality of D_k and δ_k for $k>2$. We show here that $D_k < \delta_k$

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as a corollary of the fact that $T(x)$ has infinitely many changes of sign; this in turn is a corollary of

THEOREM 1. *Let* $\rho_j=\frac{1}{2}+i\gamma_j$ $(j=1,\ 2)$ *denote the first two zeros of* $\zeta(s)$ *above the* σ *axis* $(\gamma_1\approx 14,\ \gamma_2\approx 21)$. *Let* $\alpha_1=\zeta(\rho_1/k)/[\rho_1\zeta'(\rho_1)]$ *and let* $L=2(1-\gamma_1/\gamma_2)|\alpha_1|>0$. *Then*

$$\liminf_{x\to\infty} x^{-1/2k}T(x) \leqq -L \quad \textit{and} \quad \limsup_{x\to\infty} x^{-1/2k}T(x) \geqq L.$$

Before we prove Theorem 1, it is convenient to introduce some notation. Let $b_n=a_n-1/\zeta(k)$, where $a_n=1$ or 0 according as n is k-free or not. Then

$$T(x) = \sum_{n\leq x} b_n, \tag{3}$$

and

$$\sum_{n=1}^{\infty} b_n n^{-s} = \frac{\zeta(s)}{\zeta(ks)} - \frac{\zeta(s)}{\zeta(k)}, \tag{4}$$

where the series converges for $\sigma>1/k$ by (1).

The proof of Theorem 1 is based on a theorem of Ingham [1]:

THEOREM. *Let*

$$F(s) = \int_0^\infty A(u)e^{-su}\,du, \tag{5}$$

where $A(u)$ *is absolutely integrable over every finite interval* $0\leqq u\leqq U$, *and the integral is convergent in some half plane* $\sigma>\sigma_1\geqq 0$.

Let $A^*(u)$ be a real trigonometric polynomial,

$$A^*(u) = \sum_{n=-N}^{N} \alpha_n \exp(it_nu)$$

$$= \alpha_0 + 2\ \mathrm{Re} \sum_{n=1}^{N} \alpha_n \exp(it_nu) \quad (t_n \text{ real}, t_{-n} = -t_n, \alpha_{-n} = \bar{\alpha}_n),$$

and let

$$F^*(s) = \int_0^\infty A^*(u)e^{-su}\,du = \sum_{n=-N}^{N} \frac{\alpha_n}{s-it_n} \qquad (\sigma > 0).$$

Suppose that $F(s)-F^*(s)$ can be continued as an analytic function throughout some domain containing the region $\sigma\geqq 0$, $-T\leqq t\leqq T$ for some fixed $T>0$. Then,

$$\liminf_{u\to\infty} A(u) \leqq \liminf_{u\to\infty} \overset{*}{A}_T(u) \leqq \limsup_{u\to\infty} \overset{*}{A}_T(u) \leqq \limsup_{u\to\infty} A(u),$$

where

$$\begin{aligned}\overset{*}{A}_T(u) &= \sum_{|t_n|<T} [1 - (|t_n|/T)]\alpha_n \exp(it_n u) \\ &= \alpha_0 + 2 \operatorname{Re} \sum_{0<t_n<T} (1 - t_n/T)\alpha_n \exp(it_n u).\end{aligned}$$

To apply Ingham's theorem, let

$$A(u) = e^{-u/2k}T(e^u).$$

From (3), (4), and (5), we see that

$$F(s) = \frac{2k}{2ks+1}\left[\frac{\zeta(s+1/2k)}{\zeta(ks+1/2)} - \frac{\zeta(s+1/2k)}{\zeta(k)}\right].$$

Let

$$A^*(u) = 2 \operatorname{Re} \sum_{n=1}^{2} \alpha_n \exp(i\gamma_n u/k), \text{ where } \alpha_n = \operatorname*{Re}_{s=i\gamma_n/k} sF(s) \quad (n = 1, 2).$$

Finally, let $T=\gamma_2/k$ so that

$$\overset{*}{A}_T(u) = 2 \operatorname{Re}\left[(1 - \gamma_1/\gamma_2)\alpha_1 \exp(i\gamma_1 u/k)\right].$$

Theorem 1 is now an immediate consequence of Ingham's theorem.

In closing, it should be noted that the gap between (1) and Theorem 1 will be hard to close; from (2) and (3), we see that $T(x) = O(x^{\sigma/k})$ for all $\sigma > 1/2$ implies that $\zeta(s)$ has no zeros in the half plane $\sigma > 1/2$. In the reverse direction, the best known result on the Riemann hypothesis is Axer's result [**4**],

$$T(x) = O(x^{(2+\epsilon)/(2k+1)}).$$

Thus, even with the Riemann hypothesis, the order of $T(x)$ remains in question.

Professor Bateman has noted that a slightly weaker form of Theorem 1 follows from Landau's theorem. Suppose that for some real c and positive α, $T(x)+cx^\alpha$ has the same sign for $x>x_0$. Then Landau's theorem applied to the formula

$$s\int_1^\infty \frac{T(x)+cx^\alpha}{x^{s+1}}\,dx = \frac{\zeta(s)}{\zeta(ks)} - \frac{\zeta(s)}{\zeta(k)} + \frac{cs}{s-\alpha}$$

says that the function on the right has no singularities in the half plane $\sigma > \alpha$ since the function is regular on the part of the σ axis with $\sigma > \alpha$. Since there is a singularity at ρ_1/k, this is a contradiction if $\alpha < 1/(2k)$. Thus $T(x) + cx^\alpha$ changes sign infinitely often for any real c and any $\alpha < 1/(2k)$.

References

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