COEFFICIENT ESTIMATES FOR STARLIKE FUNCTIONS OF ORDER α

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In [1] MacGregor obtained upper bounds for the moduli of the coefficients of a function $z+\sum_{k+1}^{\infty} a_n z^n$ which is starlike in the unit disc. The purpose of this note is to extend MacGregor's result to the class of starlike functions of order α introduced by Robertson, [2], and to obtain an improvement on this when $f(z)$ is bounded in the unit disc.

DEFINITION. A function $f(z)$ is said to be *starlike of order* α, $(0 \leqq \alpha < 1)$, if it is univalent and $\operatorname{Re}\{zf'(z)/f(z)\} \geqq \alpha$ for $|z| < 1$.

THEOREM. *If* $f(z) = z + \sum_{n=k+1}^{\infty} a_n z^n$ *is starlike of order* α *then*

$$|a_n| \leqq \binom{(2-2\alpha)/k + m - 1}{m}$$

where $mk+1 \leqq n \leqq mk+k$, $m = 1, 2, 3, \cdots$. *If, further,* $|f(z)| < 1$ *for* $|z| < 1$ *then we also have*

$$\sum_{n=p-k+1}^{p} (n+1-2\alpha)^2 |a_n|^2 \leqq 4(1-\alpha)^2 \quad \text{for } p \geqq k.$$

LEMMA 1.

$$4(1-\alpha)\left\{1-\alpha+\sum_{m=1}^{q-1}(mk+1-\alpha)\left[\frac{1}{m!}\prod_{\mu=0}^{m-1}\left(\mu+\frac{2-2\alpha}{k}\right)\right]^2\right\}$$
$$= \left\{\frac{k}{(q-1)!}\prod_{\mu=0}^{q-1}\left(\mu+\frac{2-2\alpha}{k}\right)\right\}^2 \quad \text{for } q = 2, 3, \cdots.$$

This is easily proved by induction on q.

LEMMA 2. *If* $k = 1, 2, \cdots$, $q = 1, 2, \cdots$, *and* $\alpha < 1$ *then*

$$(n-1)^2 \geqq (qk)^2(n-\alpha)/(qk+1-\alpha) \quad \text{for } n \geqq qk+1.$$

PROOF OF THEOREM. By the method of [1], if $g(z) = zf'(z)/f(z)$ and $h(z) = (g(z)-1)/(g(z)+1-2\alpha) = b_k z^k + b_{k+1}z^{k+1} + \cdots$ then $(n-1)a_n = 2(1-\alpha)b_{n-1}$ for $n = k+1$ to $2k$, and

$$\text{(i)} \qquad \sum_{n=k+1}^{2k} (n-1)^2 |a_n|^2 \leqq 4(1-\alpha)^2.$$

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Also, for some set of constants d_n,

$$\sum_{n=k+1}^{p} (n-1)a_n z^n + \sum_{n=p+1}^{\infty} d_n z^n$$
$$= h(z)\left\{2(1-\alpha)z + \sum_{n=k+1}^{p-k} (n+1-2\alpha)a_n z^n\right\}.$$

Since $|h(z)|<1$ for $|z|<1$ it then follows as in the original paper [3] of Clunie that, for $0 \leqq r < 1$,

$$\sum_{n=k+1}^{p} (n-1)^2 |a_n|^2 r^{2n} \leqq 4(1-\alpha)^2 r^2 + \sum_{n=k+1}^{p-k} (n+1-2\alpha)^2 |a_n|^2 r^{2n}.$$

Since $a_1=1$ and $a_2= \cdots =a_k=0$ this may be written as

$$\text{(ii)} \qquad \sum_{n=p-k+1}^{p} (n-1)^2 |a_n|^2 r^{2n} \leqq 4(1-\alpha) \sum_{n=1}^{p-k} (n-\alpha) |a_n|^2 r^{2n}.$$

Letting r tend to 1 yields

$$\text{(iii)} \qquad \sum_{n=p-k+1}^{p} (n-1)^2 |a_n|^2 \leqq 4(1-\alpha)\left\{1-\alpha + \sum_{n=k+1}^{p-k} (n-\alpha) |a_n|^2\right\}.$$

We next consider

$$\text{(A)} \qquad \sum_{n=mk+1}^{mk+k} (n-1)^2 |a_n|^2 \leqq \left\{\frac{k}{(m-1)!} \prod_{\mu=0}^{m-1} \left(\mu + \frac{2-2\alpha}{k}\right)\right\}^2$$

and

$$\text{(B)} \qquad \sum_{n=mk+1}^{mk+k} (n-\alpha) |a_n|^2 \leqq (mk+1-\alpha) \left\{\frac{1}{m!} \prod_{\mu=0}^{m-1} \left(\mu + \frac{2-2\alpha}{k}\right)\right\}^2.$$

In the case $m=1$, (A) reduces to equation (i) and (B) follows by an application of Lemma 2 to (A). For $m=q>1$, (A) and (B) are proved inductively, as in [1], by applying (B) with $m=1$ to $q-1$ and Lemma 1 to (iii) with $p=(q+1)k$, and Lemma 2 to (A) with $m=q$.

From (A) it follows that, for $mk+1 \leqq n \leqq mk+k$,

$$|a_n| \leqq \frac{k}{(n-1)(m-1)!} \prod_{\mu=0}^{m-1} \left(\mu + \frac{2-2\alpha}{k}\right)$$
$$\leqq \frac{1}{m!} \prod_{\mu=0}^{m-1} \left(\mu + \frac{2-2\alpha}{k}\right).$$

It should be noted that this result

$$|a_n| \leqq \binom{(2-2\alpha)/k+m-1}{m}$$

is "sharp" for $n=mk+1$, $(m=1, 2, \cdots)$, for the function $f(z) = z(1-z^k)^{-2(1-\alpha)/k}$; and that when $\alpha=0$ it gives the same bounds for the coefficients as Waadeland [4] found in the case of k-symmetric univalent functions.

We also have, from (ii), that

$$\sum_{n=p-k+1}^{p} (n+1-2\alpha)^2 |a_n|^2 r^{2n} \leqq 4(1-\alpha)\sum_{n=1}^{p} (n-\alpha)|a_n|^2 r^{2n}$$

$$\leqq 4(1-\alpha)\sum_{n=1}^{\infty} (n-\alpha)|a_n|^2 r^{2n}.$$

As on page 232 of [5], the right hand-side does not exceed

$$\frac{2(1-\alpha)}{\pi}\int_0^{2\pi}\left\{\mathrm{Re}\,\frac{zf'(z)}{f(z)}-\alpha\right\}d\theta \qquad \text{where } z = re^{i\theta}$$

$$=4(1-\alpha)^2.$$

It now follows that

$$\sum_{n=p-k+1}^{p} (n+1-2\alpha)^2 |a_n|^2 \leqq 4(1-\alpha)^2 \quad \text{for } p \geqq k.$$

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References

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