DEGREE OF APPROXIMATION BY POLYNOMIALS TO FUNCTIONS OF BOUNDED VARIATION[1]

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1. **Introduction.** Let $F(z)$ be analytic in the interior of a Jordan curve C, continuous in the closed region. For a point $z_0 \in I(C)$, $F'(z)$ is uniquely determined. However, for z_0 on C we define $F'(z_0)$ as $\lim_{z\to z_0}((F(z)-F(z_0))/(z-z_0))$, z on C provided this limit exists. Derivatives of higher orders are defined similarly. The subclass of the above functions, where $F^{(p)}(z)$ are of bounded variation on C will be designated throughout as $B(p, C)$. The concept of bounded variation is assumed. Note that if $F \in B(p, C)$, where C is a rectifiable Jordan curve, then $F \in B(q, C)$, $q=0, 1, \cdots, p-1$; also every analytic function over a rectifiable Jordan curve is of bounded variation there.

When C is an analytic Jordan curve (analytic), G. Faber [2] proved that analytic functions can be expanded in $I(C)$ by an infinite series of certain polynomials known as Faber polynomials—§3.

The object of this paper is to obtain the degree of approximation of such series to functions in $B(p, C)$ (Theorem 1). In Lemma 2, §2 an estimate of the Taylor coefficients for functions in $B(p, C)$ where C is $|z| = 1$ is obtained. This estimate is employed in obtaining an estimate on the corresponding Faber coefficients in case C is analytic (Lemma 3, §4). Lemma 4, §5 gives an estimate on the Faber polynomials associated with the analytic curve C. Both Lemmas 4 and 5 are used in proving Theorem 1. Two related results, Theorem 2 and Theorem 3, are stated in §7 but only the latter is proved.

A similar problem was solved by W. Sewell [4] in which the class of functions was subjected to different hypotheses, mainly that their pth derivatives satisfy a Lipschitz condition of order α on C, $0<\alpha\leqq 1$.

2. **Taylor coefficients.** First we mention a Corollary of D. Jackson [3, p. 50].

COROLLARY (JACKSON). *Let $F(x)$ be a real-valued function of the real variable x of period 2π. Let $F^{(p)}(x)$, $p\geqq 0$, exist and be of bounded variation over a period. Then for $k>0$*

$$|b_k| \leqq V(F^{(p)})/2\pi k^{p+1}, \qquad |c_k| \leqq V(F^{(p)})/2\pi k^{p+1},$$

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where b_k and c_k are the kth coefficients of the Fourier series corresponding to $F(x)$ on a period, i.e.,

$$F(x) \cong b_0/2 + \sum_1^\infty (b_k \cos kx + c_k \sin kx).$$

$V(G)$ means, throughout, the total variation of G.

The above corollary can be extended to include the case where $F(x)$ is a complex-valued function of the real variable x. This can be accomplished easily if the above corollary is applied to the real and the imaginary parts of $F(x)$.

LEMMA 1. *Let $F(x)$ be a complex-valued function of the real variable x of period 2π. Let $F^{(p)}(x)$, $p \geqq 0$, exist and be of bounded variation over a period. Then for $k>0$,*

$$(1) \qquad |b_k| \leqq V(F^{(p)})/\pi k^{p+1}, \qquad |c_k| \leqq V(F^{(p)})/\pi k^{p+1}.$$

LEMMA 2. *Let $F \in B(p, C)$, where C is the circle $|z|=1$, $p \geqq 0$. Let $G(\theta)=F(e^{i\theta})$, and $F(z)=\sum_0^\infty a_k z^k$. Then*

$$(2) \qquad |a_k| \leqq V(G^{(p)})/k^{p+1}.$$

PROOF. Considering $F(e^{i\theta})=G(\theta)$ as a function of θ it follows that

$$(3) \qquad \begin{aligned} \frac{b_k - ic_k}{2} &= \frac{1}{2\pi i}\int_0^{2\pi} F(e^{i\theta})(\cos k\theta - i \sin k\theta)\, d\theta \\ &= \frac{1}{2\pi i}\int_{|z|=1} \frac{F(z)}{z^{k+1}}\, dz = a_k, \end{aligned}$$

where b_k and c_k are the Fourier coefficients corresponding to $G(\theta)$. Since $F \in B(p, C)$, C is $|z|=1$, then by chain differentiation of $G(\theta) = F(e^{i\theta})$, $G^{(p)}(\theta)$ is of bounded variation over C. Combining (1) and (3) yields (2).

3. **The Faber theorem.** Let C be an analytic curve in the z-plane and $A(C)$ be its exterior. There is a unique analytic function $f(z)$ which maps $A(C)$ in a 1-1 manner onto the exterior of a circle $|w|=\rho$, designated as K_ρ, such that the points at infinity correspond to each other and its power series about $z=\infty$ has the normalization $w=f(z)=z+d_0+d_1/z+\cdots$. Because $f(z)$ is schlicht there exists a unique inverse schlicht function $g(w)$ in $A(K_\rho)$ whose power series expansion about $w=\infty$ is $z=g(w)=w+e_0+e_1/w+\cdots$, with $\limsup |e_n|^{1/n}=\bar{\rho}$, $\bar{\rho}<\rho$.

Because of the analyticity of C, there is a minimum number ρ_0,

$\bar{\rho} \leqq \rho_0 < \rho$, for which $g(w)$ is still schlicht in $A(K_{\rho_0})$. Let C_{ρ_0} be the level curve in the z-plane corresponding to the circle K_{ρ_0} under $w=f(z)$. Note that C is C_ρ. The Faber polynomials $\{f_n(z)\}$, $n=0, 1, 2, \cdots$ are the polynomial parts of the formal expansion of $(f(z))^n$ about $z=\infty$. In the sequel $f(z)$, $g(w)$ and $f_n(z)$ will be used in the above contexts. It is easily seen that

$$f_n(z) = \frac{1}{2\pi i}\int_{C_r} \frac{(f(t))^n}{t-z}\,dt, \tag{4}$$

where $z \in I(C_r)$ with an appropriate choice $r > \rho_0$; see e.g. [1, p. 6].

Faber proved the following theorem.

THEOREM (FABER). *Let $F(z)$ be analytic in $I(C)$, C an analytic curve. For $z \in I(C)$*

$$F(z) = \sum_0^\infty a_k f_k(z), \tag{5}$$

where $\{f_k(z)\}$ are the Faber polynomials associated with C. Here

$$a_k = \frac{1}{2\pi i}\int_{K_{\rho_1}} \frac{F(g(w))}{w^{k+1}}\,dw, \tag{6}$$

for $\rho_0 < \rho_1 < \rho$. The series in (5) converges uniformly in any closed subset of $I(C)$.

The coefficients in (5) are referred to as *Faber coefficients.*

4. **Faber coefficients.** The following lemma provides an estimate on Faber coefficients.

LEMMA 3. *Let $F \in B(p, C)$, C an analytic curve. Then for $k > 0$, $p \geqq 0$,*

$$|a_k| = V(G^{(p)})/\rho^k k^{p+1} \tag{7}$$

where $G(\theta) = \sum_0^\infty (a_k\rho^k)e^{ik\theta}$, $0 \leqq \theta \leqq 2\pi$, and $\{a_k\}$ are the Faber coefficients associated with $F(z)$ in $I(C_\rho)$.

PROOF. Since $g(w)$ is schlicht, $F(g(w))$ is analytic in $\rho_0 < \rho' \leqq |w| < \rho$ and continuous in the closure of this annular ring. It is clear from (6) that the Laurent series of $F(g(w))$ in $\rho_0 < \rho' \leqq |w| < \rho$ is

$$F(g(w)) = \sum_0^\infty a_k w^k + \sum_1^\infty A_{-k}/w^k, \tag{8}$$

where a_k are the Faber coefficients associated with $F(z)$ in $I(C_\rho)$. Let $\alpha(w) = -\sum_1^\infty A_{-k}/w^k$, $\beta(w) = F(g(w))$. Define

(9) $\lambda(w) = \alpha(w) + \beta(w)$ for w on K_ρ and $\lambda(w) = \sum_0^\infty a_k w^k$ for $|w| < \rho$.

Thus $\lambda(w)$ is analytic in $I(K_\rho)$ and continuous in $\bar{I}(K_\rho)$. From (8) and (9) $\lambda(w) = \alpha(w) + \beta(w)$, $\rho' \leqq |w| \leqq \rho$. Let $w = \rho y$. Then $\lambda(w) = \lambda(\rho y) = \Phi(y)$ for $|y| \leqq 1$ and

$$\Phi(y) = \sum_0^\infty (a_k \rho^k) y^k, \qquad |y| < 1. \tag{10}$$

By chain differentiation of $\beta(w)$ where $z = g(w)$ it is readily seen that $\beta^{(p)}(w)$ is of bounded variation on K_ρ. Thus $\lambda \in B(p, K_\rho)$ and since $\Phi^{(p)}(y) = \rho^p \lambda^{(p)}(w)$, $\Phi \in B(p, K_1)$. Let $G(\theta) = \Phi(e^{i\theta})$ for $0 \leqq \theta \leqq 2\pi$.

Since $G(\theta)$ is periodic of period 2π, continuous and of bounded variation over the interval $[0, 2\pi]$, it follows from [**6**, pp. 175–180] that $G(\theta) = \sum_0^\infty (a_k \rho^k) e^{ik\theta}$. From (2) and (10) it follows $|a_k \rho^k| \leqq V(G^{(p)})/k^{p+1}$, thus $|a_k| \leqq V(G^{(p)})/\rho^k k^{p+1}$.

5. **Faber polynomials.** The following lemma provides an estimate on Faber polynomials.

LEMMA 4. *Let C be an analytic curve. Then*

$$f_k(z) = (f(z))^k (1 + G_{\rho_1} \cdot \theta_k(z)) \tag{11}$$

for $z \in A(C_{\rho_1})$ and ρ_1 a fixed number satisfying $\rho_1 > \rho_0$; G_{ρ_1} is a real constant independent of k and z but dependent on ρ_1. The function $\theta_k(z)$ is analytic in $\bar{A}(C_{\rho_1})$ and $|\theta_k(z)| \leqq (\rho_2/\rho_1)^K$ for $z \in \bar{A}(C_{\rho_1})$, $k > 0$, where ρ_2 is a definite fixed number with $\rho_0 < \rho_2 < \rho_1$. Also for z on $C_{\rho'}$, $\rho' \geqq \rho_1$

$$|f_k(z)| < (1 + G_{\rho_1}) \rho'^k. \tag{12}$$

PROOF. Let ρ_1 be a fixed number such that $\rho_1 > \rho_0$ and let $\rho_2 = (\rho_1 + \rho_0)/2$. For an arbitrary but fixed point $z \in A(C_{\rho_2})$, choose ρ_3 large enough, so that $z \in I(C_{\rho_3})$ and formula (4) holds for $r = \rho_3$. Thus

$$f_k(z) = \frac{1}{2\pi i} \int_{C_{\rho_3}} \frac{(f(t))^k}{t - z} dt = (f(z))^k + \frac{1}{2\pi i} \int_{C_{\rho_2}} \frac{(f(t))^k}{t - z} dt.$$

We put the above in a different form

$$f_k(z) = (f(z))^k (1 + I_k(z)),$$

where

$$I_k(z) = \frac{1}{2\pi i} \int_{C_{\rho_2}} \left(\frac{f(t)}{f(z)}\right)^k \frac{1}{t - z} dt.$$

Note that the maximum of $|I_k(z)|$ is attained on C_{ρ_1} since the function is analytic in the closed exterior of the curve and $\lim_{z\to\infty} I_k(z)=0$. Thus for $z\in\overline{A}(C_{\rho_1})$,

$$\max |I_k(z)| \leqq \frac{1}{2\pi}\left(\frac{\rho_2}{\rho_1}\right)^k\cdot\frac{L}{d(C_{\rho_1}, C_{\rho_2})},$$

where L is the length of C_{ρ_2} and $d(C_{\rho_1}, C_{\rho_2})$ is the minimum distance between C_{ρ_1} and C_{ρ_2}. Let

$$G_{\rho_1} = \frac{L}{2\pi d(C_{\rho_1}, C_{\rho_2})} \quad \text{and} \quad \theta_k(z) = \frac{I_k(z)}{G}.$$

Then formula (11) follows from substituting for $I_k(z)$ in the equation of $f_k(z)$ above. For z on $C_{\rho'}$, $\rho'\geqq\rho_1$ it follows that $|\theta_k(z)| \leqq(\rho_2/\rho_1)^k<1$. Thus formula (12) is proved.

6. **The main theorem.** Lemmas 3 and 4 yield:

THEOREM 1. *Let $F\in B(p, C)$, where C is an analytic curve and $p\geqq 1$. Then for $z\in\overline{I}(C)$, $n>0$,*

$$\max\left|F(z) - \sum_0^n a_k f_k(z)\right| \leqq \frac{Q\cdot V(G^{(p)})}{pn^p}, \tag{13}$$

where $\sum_0^\infty a_k f_k(z)$ is the usual Faber series for $F(z)$ in $I(C)$ and Q is a constant independent of p, n and z but dependent on ρ, and $V(G^{(p)})$ is the total variation of $G(\theta)$ on $[0, 2\pi]$ and $G(\theta)=\sum_0^\infty (a_k\rho^k)e^{ik\theta}$.

PROOF. First we will show that the Faber series converges to $F(z)$ in $\overline{I}(C)$, provided $F\in B(p, C)$, where C is an analytic curve—see §3. Consider

$$|S_{n+m}(z) - S_n(z)| = \left|\sum_{n+1}^{n+m} a_k f_k(z)\right| \leqq \sum_{n+1}^{n+m} |a_k|\,|f_k(z)|$$

for z on C, where $S_p(z)=\sum_0^p a_k f_k(z)$. Let ρ be ρ_1 in Lemma 4. Then (7) and (12) for z on C yield

$$\begin{aligned}\max|S_{n+m}(z) - S_n(z)| &\leqq \sum_{k=n+1}^{k=n+m}\frac{V(G^{(p)})}{\rho^k k^{p+1}}(1+G_\rho)\rho^k \\ &\leqq \frac{(1+G_\rho)V(G^{(p)})}{\pi}\int_n^\infty\frac{du}{u^{p+1}} \\ &= \frac{(1+G_\rho)V(G^{(p)})}{\pi pn^{p+1}}.\end{aligned}$$

By the maximum principle, for $z \in \bar{I}(C)$,

$$|S_{n+m}(z) - S_n(z)| \leqq \frac{(1+G_\rho)V(G^{(p)})}{\pi p n^{p+1}} \to 0, \quad \text{as } n \to \infty.$$

By Cauchy's convergence principle $\{S_n(z)\}$ converges uniformly to a function analytic in $I(C)$ and continuous in $\bar{I}(C)$, say $H(z)$. Let $K(z) = F(z) - H(z)$. Since $F(z) \equiv H(z)$ in $I(C)$, it follows that $K(z) \equiv 0$ in $I(C)$ and continuous in $\bar{I}(C)$. Let z_0 be any fixed point on C. Since C is analytic Jordan curve and $K(z)$ is continuous in $\bar{I}(C)$, it follows that $\lim_{z \to z_0} K(z) = K(z_0) = 0$ for $z \in I(C)$. Thus $K(z) \equiv 0$ in $\bar{I}(C)$ which implies $S_n(z) \to F(z)$ uniformly in $\bar{I}(C)$. For z on C

$$\begin{aligned} \max \left| F(z) - \sum_0^n a_k f_k(z) \right| &\leqq \max \sum_{n+1}^\infty |a_k| \cdot |f_k(z)| \\ &\leqq \frac{(1+G_\rho)V(G^{(p)})}{\pi} \sum_{n+1}^\infty \frac{1}{k^{p+1}} \\ &\leqq \frac{(1+G_\rho)V(G^{(p)})}{\pi} \int_n^\infty \frac{du}{u^{p+1}} \\ &= \frac{QV(G^{(p)})}{pn^p}, \end{aligned}$$

where $Q = (1+G_\rho)/\pi$. By the maximum principle (13) is achieved and so the main theorem is proved.

7. **Related results.** The method used in proving Theorem 1 can be employed again to show:

THEOREM 2. *Let $F \in B(p, C)$, where C is an analytic curve and $p \geqq 1$. Let $\sum_0^\infty a_k f_k(z)$ be the Faber series of $F(z)$ in $I(C)$. Let ρ_1, be a fixed number satisfying $\rho_0 < \rho_1 < \rho$. Then for $n > 0$ and $z \in \bar{I}(C_{\rho_1})$*

$$\max \left| F(z) - \sum_0^n a_k f_k(z) \right| < \frac{Q_1 V(G^{(p)})}{pn^p} \cdot \left(\frac{\rho_1}{\rho}\right)^{n+1},$$

where Q_1 is a constant independent of p, n, and z in $I(C_{\rho_1})$ but dependent on ρ_1 and $G(\theta) = \sum_0^\infty (a_k \rho^k) e^{ik\theta}$, for θ on $[0, 2\pi]$.

THEOREM 3. *Assume the same hypotheses as for the previous theorem. Let $P_n(z)$ be the polynomial of degree n found by interpolation to $F(z)$ in the roots of the Faber polynomial $f_{n+1}(z)$. Then for n sufficiently large and $z \in \bar{I}(C_{\rho_1})$*

$$\max |F(z) - P_n(z)| \leqq \frac{Q_2 V(G^{(p)})}{pn^p} \cdot \left(\frac{\rho_1}{\rho}\right)^{n+1},$$

where Q_2 is a constant independent of p, n and z but dependent on ρ_1.

The following Theorem due to Sewell and Walsh [5] is needed in the proof of Theorem 3.

THEOREM (SEWELL AND WALSH). *Let $F(z)$ be an analytic function in the interior of an analytic curve C and continuous in its closure. Let z_i, $i=1, 2, \cdots, n+1$ be in $I(C)$ and $R_{n+1}(z)-\prod^{n+1}(z-z_i)$. Let $P_n(z)$ be the polynomial of degree n found by interpolation to $F(z)$ in the points z_i, $i=1, 2, \cdots, n+1$. Then*

$$F(z) - P_n(z) = \frac{1}{2\pi i}\int_C \frac{R_{n+1}(z)}{R_{n+1}(t)}\cdot\frac{F(t) - S_n(z)}{t - z}\,dt, \tag{14}$$

where $S_n(z)$ is an arbitrary polynomial of degree n.

PROOF OF THEOREM 3. Let $\rho_2=(\rho_1+\rho_0)/2$. The first part of Lemma 4 yields

$$\max_{z\in\bar{A}(C)}\left|\frac{f_{n+1}(z)}{(f(z))^{n+1}} - 1\right| \leqq G_{\rho_1}(\rho_2/\rho_1)^{n+1}.$$

Thus there exists a positive integer n_0 such that for $n \geqq n_0$

$$\max_{z\in\bar{A}(C)}\left|\frac{f_{n+1}(z)}{(f(z))^{n+1}} - 1\right| < 1/2,$$

which implies that for such sufficiently large n, $f_{n+1}(z)$ has no zeros in $\bar{A}(C)$. The Sewell and Walsh Theorem is applicable now when z_i, $i=1, 2, \cdots, n+1$, are chosen to be the zeros of $f_{n+1}(z)$, $n \geqq n_0$, $R_{n+1}(z)$, $S_n(z)=\sum_0^n a_k f_k(z)$. Hence (14) becomes

$$F(z) - P_n(z) = \frac{1}{2\pi i}\int_C \frac{f_{n+1}(z)}{f_{n+1}(t)}\cdot\frac{F(z) - \sum_0^n a_k f_k(t)}{t - z}\,dt \tag{15}$$

for $z\in I(C)$. For z on C formula (11) yields

$$\min|f_{n+1}(z)| \geqq \rho^{n+1}|1 - G_{\rho_1}\theta_{k+1}(z)| \geqq \rho^{n+1}/2$$

for $n \geqq n_0$. The above lower bound and (12) yield

$$\max\left|\frac{f_{n+1}(z)}{f_{n+1}(t)}\right| \leqq 2(1 + G_{\rho_1})(\rho_1/\rho)^{n+1}, \tag{16}$$

for $n \geqq n_0$ where the maximum is taken over t on C and z on C_{ρ_1}. Formulas (15) and (16) and Theorem 1 yield

$$\max|F(z) - P_n(z)| \leqq 2(1 + G_{\rho_1})(\rho_1/\rho)^{n+1}\cdot\frac{QLV(G^{(p)})}{2\pi\delta p n^p},$$

where L is the length of C, δ is the minimum distance between C and C_{ρ_1}. Let $Q_2=QL(1+G_{\rho_1})/\pi\delta$. Then Theorem 3 is concluded.

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