ON HIGH INDICES THEOREMS

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1. A series

$$\sum_{n=1}^{\infty} a_n$$

is said to be lacunary if all its terms are zero, except perhaps for a set of indices

$$0 < n_1 < n_2 < \cdots$$

which satisfy the condition

$$n_{i+1}/n_i \geqq q > 1, \qquad i = 1, 2, \cdots.$$

Throughout, let $\{\lambda_n\}$ be a sequence of positive numbers such that

$$1 \leqq \lambda_1 < \lambda_2 < \cdots$$

and let $\sum a_n$ be the given infinite series.

The series $\sum a_n$ is said to be summable (A, λ) if

$$f(x) = \sum a_n \exp[-\lambda_n x] \tag{1.1}$$

converges for $x > 0$ and $\lim f(x)$ as $x \to 0$ exists and is finite.

The Dirichlet series (1.1) is called lacunary if the λ_n satisfy the condition

$$\lambda_{n+1}/\lambda_n \geqq q > 1, \qquad n = 1, 2, \cdots. \tag{1.2}$$

The series $\sum a_n$ is called $|A, \lambda|$ summable if the series (1.1) converges for $x > 0$ and $f(x)$ is of bounded variation in $(0, \infty)$.

We write

$$\begin{aligned} A_\lambda^k(x) &= \sum_{\lambda_n < x} (x - \lambda_n)^k a_n \\ &= \int_1^x (x - t)^k dA_\lambda(t), \end{aligned}$$

$$A_\lambda^0(x) = A_\lambda(x) = \sum_{\lambda_n < x} a_n,$$

$$A_\lambda^k(x) = 0 \quad \text{for } x \leqq 1 \text{ and } k > -1.$$

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We also write

$$B_{\lambda}^{k}(x) = \sum_{\lambda_n < x} (x - \lambda_n)^k \lambda_n a_n.$$

The series $\sum a_n$ is said to be summable (R, λ, k) to the sum s, if $\lim x^{-k} A^k(x) = s$ as $x \to \infty$; the series is said to be absolutely Riesz summable with index m, or simply $|R, \lambda, k|_m$ summable if

$$\int_1^\infty x^{m-1} \left| \frac{d}{dx} x^{-k} A_\lambda^k(x) \right|^m dx < \infty$$

where $k > 0$, $m \geqq 1$, and $km' > 1$ $(1/m + 1/m' = 1)$. The first theorem of consistency for $|R, \lambda, k|_m$ summability has been proved by Mazhar [4].

We say that the given series $\sum a_n$ is summable $|R, \lambda, k, \gamma|_m$ if

$$\int_1^\infty x^{m\gamma + m - 1} \left| \frac{d}{dx} x^{-k} A_\lambda^k(x) \right|^m dx < \infty$$

where $k > 0$, $km' > 1$, $k > \gamma - 1$ and γ is a real number.

$|R, \lambda, k, 0|_m$ summability is the same as $|R, \lambda, k|_m$ summability.

2. The Hardy-Littlewood "high indices" theorem [1] asserts that for a lacunary series Abel summability implies convergence. Zygmund [6] has shown that if $\sum a_n$ is summable $|A, \lambda|$ and the λ_n satisfy (1.2) then $\sum a_n$ is absolutely convergent.

Waterman [5] generalized Zygmund's result and proved the following theorems.

THEOREM A. *If the series* $f(x) = \sum a_n \exp[-\lambda_n x]$ *is lacunary*, $m > 1$, *and*

$$\int_0^\infty (1 - e^{-x})^{m-1} |f'(x)|^m dx < \infty$$

then

$$\sum_{n=1}^\infty |a_n|^m < \infty.$$

THEOREM B. *If the series* $f(x) = \sum a_n \exp[-\lambda_n x]$ *is lacunary*, $m > 1$, $1 \leqq \beta \leqq m$, *and*

$$\int_0^\infty (1 - e^{-x})^{\beta-1} |f'(x)|^m dx < \infty$$

then

$$\sum_{n=1}^{\infty} | a_n |^m \lambda_n^{m-\beta} < \infty .$$

The following theorem is due to Hardy and Riesz [2].

THEOREM C. *If* $\sum a_n$ *is summable* (R, λ, k) *and* λ_n*'s satisfy* (1.2), *then* $\sum a_n$ *converges.*

3. We prove the following theorems.

THEOREM 1. *If* (i) $\sum a_n$ *is summable* $|R, \lambda, k|_m$,
(ii) $f(x) = \sum a_n \exp[-\lambda_n x]$ *converges for* $x > 0$, *and*
(iii) *the* λ_n *satisfy* (1.2), *then* $\sum_1^\infty |a_n|^m < \infty$.

THEOREM 2. *If* $\sum a_n$ *is summable* $|R, \lambda, k, \gamma|_m$, $0 < \gamma \leqq 1 - 1/m$, *and the* λ_n *satisfy* (1.2), *then* $\sum |a_n|^m \lambda_n^{m\gamma} < \infty$.

I wish to thank Professor Waterman for suggesting the problem and for his valuable guidance.

3.1. The following lemmas will be used to prove our theorems.

LEMMA 1 [3]. *If* $B^k(x)$ *is the* (R, λ, k) *sum of the series* $\sum a_n \lambda_n$, *then for* $k > 0$

$$\frac{d}{dx}\left(x^{-k} A_\lambda^k(x)\right) = k x^{-k-1} B_\lambda^{k-1}(x).$$

LEMMA 2 [2], [3]. *If* $k > -1$, $p > 0$, *then*

$$A_\lambda^{k+p}(x) = \frac{\Gamma(k+p+1)}{\Gamma(k+1)\Gamma(p)} \int_1^x (x-t)^{p-1} A_\lambda^k(t)\, dt.$$

LEMMA 3. *If* $\sum a_n$ *is summable* $|R, \lambda, k|_m$, *then it is also summable* $|R, \lambda, h|_m$ *for* $h > k$.

PROOF OF LEMMA 3. Summability $|R, \lambda, k|_m$ of $\sum a_n$ with Lemma 1 implies

$$\int_1^\infty x^{-mk-1} | B_\lambda^{k-1}(x) |^m \, dx < \infty,$$

and to prove the lemma it is sufficient to show that

$$\int_1^\infty x^{-mh-1} | B_\lambda^{h-1}(x) |^m \, dx < \infty.$$

Let $h = k + p$, $p > 0$. Applying Lemma 2 to the series $\sum a_n \lambda_n$ we have

$$B_\lambda^{h-1}(x) = M \int_1^x (x-t)^{p-1} B_\lambda^{k-1}(t)\, dt.$$

Throughout this paper M denotes a positive constant which is not necessarily the same at every occurrence.

Applying Hölder's inequality, we have

$$
\begin{aligned}
|B_\lambda^{h-1}(x)|^m &\leqq M \left\{\int_1^x (x-t)^{p-1} | B_\lambda^{k-1}(t)|^m dt\right\} \left\{\int_1^x (x-t)^{p-1} dt\right\}^{m-1} \\
&< Mx^{(m-1)p} \int_1^x (x-t)^{p-1} | B_\lambda^{k-1}(t)|^m dt.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&\int_1^\infty x^{-mh-1} | B_\lambda^{h-1}(x)|^m dx \\
&\quad < M \int_1^\infty x^{-mk-p-1} dx \int_1^x (x-t)^{p-1} | B_\lambda^{k-1}(t)|^m dt \\
&\quad = M \int_1^\infty | B_\lambda^{k-1}(t)|^m dt \int_t^\infty (x-t)^{p-1} x^{-mk-p-1} dx \\
&\quad = M \int_1^\infty t^{-mk-1} | B_\lambda^{k-1}(t)|^m dt \\
&\quad < \infty.
\end{aligned}
$$

LEMMA 4. *Let* $\gamma > \mu$, $m > p \geqq 1$. *If* $\sum a_n$ *is summable* $|R, \lambda, k, \gamma|_m$ *then it is also summable* $|R, \lambda, k, \mu|_p$.

PROOF OF LEMMA 4. Under the hypothesis of the lemma we have to show that

$$
I = \int_1^\infty x^{p\mu - pk - 1} | B_\lambda^{k-1}(x)|^p dx < \infty.
$$

Using Hölder's inequality with indices m/p and $m/(m-p)$ we have

$$
I \leqq \left\{\int_1^\infty x^{m\gamma - mk - 1} | B_\lambda^{k-1}(x)|^m dx\right\}^{p/m} \left\{\int_1^\infty x^{-1-\epsilon} dx\right\}^{1-p/m}
$$

where

$$
\epsilon = \frac{pm(\gamma - \mu)}{m - p} > 0
$$

and the conclusion follows immediately.

LEMMA 5 [2]. *If* $f(x) = \sum a_n \exp[-\lambda_n x]$ *converges for* $x > 0$ *then*

$$f(x) = Mx^{k+1}\int_1^\infty A_\lambda^k(t)e^{-xt}\,dt.$$

4. Proof of Theorem 1. From Lemma 3 we have summability $|R, \lambda, k|_m$ of $\sum a_n$ implies its summability $|R, \lambda, k+1|_m$. Thus we have

$$\int_1^\infty x^{-mk-m-1}\,|\,B_\lambda^k(x)\,|^m\,dx < \infty. \tag{4.1}$$

Applying Lemma 5 to the series $f'(x) = -\sum a_n\lambda_n \exp[-\lambda_n x]$ we have

$$f'(x) = -Mx^{k+1}\int_1^\infty B_\lambda^k(t)e^{-xt}\,dt.$$

Let

$$\begin{aligned} I &= \int_0^\infty (1-e^{-x})^{m-1}\,|\,f'(x)\,|^m\,dx \\ &< \int_0^\infty (e^x-1)^{m-1}\,|\,f'(x)\,|^m\,dx. \end{aligned}$$

Thus

$$I < M\int_0^\infty (e^x-1)^{m-1}x^{km+m}\,dx\left|\int_1^\infty B^k(t)e^{-xt}\,dt\right|^m. \tag{4.2}$$

Let us choose p such that $1<p<m/(m-1)$; this is possible since $m>1$. Let $1/p+1/p'=1$.

Applying Hölder's inequality to t-integral of (4.2) we have,

$$\begin{aligned} I &< M\int_0^\infty (e^x-1)^{m-1}x^{km+m}\,dx\left\{\int_1^\infty |\,B_\lambda^k(t)\,|^m e^{-mxt/p}\,dt\right\} \\ &\quad\cdot\left\{\int_1^\infty e^{-m'xt/p'}\,dt\right\}^{m-1} \\ &\leqq M\int_0^\infty (e^x-1)^{m-1}x^{km+1}\,dx\int_1^\infty |\,B_\lambda^k(t)\,|^m e^{-mxt/p}\,dt \\ &= M\int_1^\infty |\,B_\lambda^k(t)\,|^m\,dt\int_0^\infty (e^x-1)^{m-1}x^{km+1}e^{-mxt/p}\,dx \\ &= M\int_1^\infty |\,B_\lambda^k(t)\,|^m t^{-km-2}\,dt\int_0^\infty (e^{x/t}-1)^{m-1}x^{km+1}e^{-mx/p}\,dx. \end{aligned}$$

Since for $t \geqq 1$, $e^{x/t}-1 \leqq e^x/t$, we have

$$I < M \int_1^\infty t^{-mk-m-1} \mid B_\lambda^k(t) \mid^m dt \int_0^\infty x^{km+1} e^{-(m/p-m+1)x} dx.$$

The x-integral converges since $m/p-m+1>0$. This together with (4.1) implies $I<\infty$.

Thus all the conditions for Theorem A are satisfied: the conclusion follows.

REMARK. For $m=1$, i.e., when $\sum a_n$ is summable $|R, \lambda, k|$, condition (ii) of Theorem 1 is redundant. In this case summability $|R, \lambda, k|$ obviously implies summability (R, λ, k) and if the λ_n's satisfy (2.1) then by Theorem C, $\sum a_n$ converges.

PROOF OF THEOREM 2. The proof is analogous to that of Theorem 1, except that the condition (ii) of Theorem 1 may be omitted. This is justified by Lemma 4 and the remark above, and the conclusion follows from Theorem B of Waterman.

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