ON EXTENSIONS OF CAYLEY ALGEBRAS

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Kaplansky in Theorem 2 of [**3**] has shown that if A is an alternative algebra with identity element 1 which contains a subalgebra B isomorphic to a Cayley algebra and if 1 is contained in B then A is isomorphic to the Kronecker product $B \otimes T$, where T is the center of A. Jacobson in Theorem 2 of [**2**] has shown that if A is an alternative algebra which contains a subalgebra B isomorphic to a Cayley algebra, then the identity e of B must lie in the center of A, provided A has characteristic different from 2. He also has given a new proof of the Kaplansky result, using his classification of completely reducible alternative bimodules. In the present note we present a generalization of the aforesaid result by Jacobson, which incidentally is also valid for characteristic 2.

THEOREM. *Let A be an alternative algebra over F and B any subalgebra with identity e. Then consider the following two conditions.*

(i) *There exist x, y in B, α in F such that $e=\alpha(x, y)^4$, where $(x, y) = xy - yx$.*

(ii) *The ideal I of B, generated by all associators of B equals B. If B satisfies* (i) *then e must be in the nucleus N of A. If B satisfies* (i) *and* (ii) *then e must be in the center C of A.*

PROOF. It will be helpful to recall some identities that hold in all alternative rings R. Let p, q, r, s, t, x, y, z be arbitrary elements of R and n an arbitrary element of the nucleus N' of R. Then

(1) $(s, t)^4$ is in N',

(2) (n, r) is in N',

(3) $(n, (x, y, z))=0$,

(4) $(n, r)(x, y, z) = -(n, x)(r, y, z)$,

(5) $(p^2, q)=p(p, q)+(p, q)p$.

A proof of (1) may be found in Theorem 3.1 (ii) of [**5**]. Proofs of (2), (3) and (4) are contained in Lemma 2.3 (ii), (iii) and (iv) of [**4**]. Identity (5) may be verified directly by expanding both sides of the equation and using the alternative law. If B satisfies the hypothesis and condition (i), then one may apply (1) directly to obtain that e

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belongs to N. If B also satisfies condition (ii), then select $n=e$, r as arbitrary in A and x, y, z arbitrary in B and substitute this in (4). Then $(e, r)(x, y, z)=-(e, x)(r, y, z)=0$, since $(e, x)=0$. The associator ideal I of B may be characterized as the additive subgroup of B generated by all elements of the form (B, B, B) and $(B, B, B)B$. We have already proved that $(e, r)(B, B, B)=0$. But (e, r) belongs to N as a result of (2), so that $(e, r)\cdot(B, B, B)B=0$ is also obvious and hence $(e, r)I=0$. Since $I=B$ and e itself belongs to B, we have $(e, r)e=0$. Using (2) we may substitute $n=(e, r)$ in (3) to obtain also that $(B, B, B)(e, r)=0$. As I may also be characterized as the additive subgroup generated by elements of the form (B, B, B) and $B(B, B, B)$, we obtain $I(e, r)=0$, and hence $e(e, r)=0$. At this point we substitute $p=e$, $q=r$ in (5) and obtain $(e, r)=(e^2, r)=e(e, r)+(e, r)e=0$. This places e in C and the proof of the theorem is complete.

Condition (i) certainly holds when B is taken to be a quaternion algebra and hence a priori if B is a Cayley algebra. Since Cayley algebras are simple and not associative, condition (ii) clearly holds when B is taken to be a Cayley algebra. Thus we obtain Jacobson's result as a corollary to our theorem. On the other hand one may readily construct other alternative algebras to which our theorem applies.

We conclude with an example that shows a quaternion algebra may be embedded as a subalgebra of an associative algebra and with the identity quaternion not in the center of the larger algebra. Consider the free associative algebra S on the four generators w, x, y, z. Define relations on x, y, z which make them behave as the quaternions $1, i, j$ respectively. In the quotient algebra R, words have the form

$$\cdots q_r w^{k_r} \cdots q_s w^{k_s} \cdots$$

where $q_i=\pm x, \pm y, \pm z, \pm yz$. Then R contains a copy of the quaternions with identity x, but $wx \neq xw$, so that x is not in the center of R. If an example that is alternative but not associative is desired, then one may take a direct product of R with a Cayley algebra.

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A CONDITION FOR A FINITE GROUP TO BE NILPOTENT

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Let $\mathcal{H}$ be a class of groups such that:

(i) If G is in $\mathcal{H}$, then every homomorphic image of G is in $\mathcal{H}$.

(ii) If G is finite and $G/\phi(G)$ is in $\mathcal{H}$, where $\phi(G)$ is the Frattini subgroup of G, then G is in $\mathcal{H}$.

Examples of such classes are the class of nilpotent groups and the class of supersolvable groups. Others can be found in a paper by Baer [1].

In this note a theorem of P. Hall on nilpotent groups is proved as a corollary to the following:

THEOREM. *If G is a finite group with a subgroup H such that $\phi(H)$ is normal in G and $G/\phi(H)$ is in $\mathcal{H}$, then G is in $\mathcal{H}$.*

LEMMA (HUPPERT). *Let G be a finite group, H be a subgroup of G, and N be a subgroup of H such that N is normal in G and $N \leqq \phi(H)$. Then $N \leqq \phi(G)$.*

PROOF. If not, G would have to have a maximal subgroup U such that $N \nleqq U$. Then $H = G \cap H = NU \cap H = N(U \cap H) = U \cap H$, since $N \leqq \phi(H)$. But this implies $H \leqq U$, contrary to $N \nleqq U$.

PROOF OF THEOREM. An application of the Lemma with $N = \phi(H)$ shows that $\phi(H) \leqq \phi(G)$. Hence $G/\phi(G)$ is in $\mathcal{H}$, and so G is in $\mathcal{H}$.

COROLLARY. *If G is a finite group with a normal subgroup H such that H is nilpotent and G/H' is nilpotent, where H' is the commutator subgroup of H, then G is nilpotent.*

PROOF. Since H is nilpotent, $\phi(H)$ contains H'. Hence $G/\phi(H)$ is

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