ON THE MULTIPLICATION OF TENSOR FIELDS

W. GREUB AND E. STAMM[1]

Let M be a paracompact n-dimensional manifold of class C^{k+1}, $T(x)$ the tangent space of the point $x \in M$ and $T(x)^*$ the dual space. A real valued $(p+r)$-linear function $\Phi(x)$ with p arguments in $T(x)$ and r arguments in $T(x)^*$ is called a *tensor of type* (p, r) at the point x. The tensors of type (p, r) at x form a linear space $T_p^r(x)$. The products of two tensors $\Phi \in T_p^r(x)$ and $\Psi \in T_q^s(x)$ is defined by

$$
\begin{aligned}
(\Phi\Psi)(x; \xi_1 \cdots \xi_{p+q}, \xi^{*1} \cdots \xi^{*r+s}) \\
&= \Phi(x; \xi_1 \cdots \xi_p, \xi^{*1} \cdots \xi^{*r}) \\
&\quad \cdot \Psi(x; \xi_{p+1} \cdots \xi_{p+q}, \xi^{*r+1} \cdots \xi^{*r+s}), \\
&\quad \xi_\nu \in T(x), \ \xi^{*\mu} \in T(x)^*.
\end{aligned}
$$

A *tensor field* of type (p, r) and class C^k on M is an assignment of tensors of type (p, r) to the points of M such that the components with respect to a local coordinate system are C^k-functions. The set of all tensor fields of type (p, r) and class C^k is a module T_p^r over the ring F of C^k-functions on M. The multiplication of tensors induces a multiplication of tensor fields in an obvious way. Now consider the F-bilinear mapping

$$T_p^r \times T_q^s \to T_{p+q}^{r+s}$$

which is defined by the multiplication. This bilinear mapping induces a F-linear mapping

$$h\colon T_p^r \otimes T_q^s \to T_{p+q}^{r+s}$$

such that

$$h(\Phi \otimes \Psi) = \Phi \cdot \Psi.$$

We shall prove in this paper the following

THEOREM. *$h\colon T_p^r \otimes T_q^s \to T_{p+q}^{r+s}$ is an isomorphism.*

For the sake of simplicity we restrict ourselves to covariant tensor

Received by the editors October 4, 1965.

[1] The second author holds a fellowship of the National Research Council of Canada.

fields, i.e. tensor fields of type $(p, 0)$ and write T_p instead of T_p^o. However, the argument can be carried over word by word to the general case.

Before giving the proof of the theorem we state some corollaries and show how they can be deducted from the theorem. We are indebted to the referee for the suggestion to include the Corollaries 3 and 4.

COROLLARY 1. *Let $(T_p^r)^*$ be the dual of the F-module T_r^p. Then*

$$(T_p^r)^* \otimes (T_q^s)^* \cong (T_p^r \otimes T_q^s)^*.$$

PROOF. There exists a canonical F-isomorphism

$$\phi_p^r\colon (T_p^r)^* \to T_r^p.$$

(cf. Helgason, *Differential geometry and symmetric spaces*, Academic Press, New York, 1958; p. 15). Since all maps in the diagram

$$\begin{array}{ccccc}
(T_p^r)^* \otimes (T_q^s)^* & \xrightarrow{\phi_p^r \otimes \phi_q^s} & T_r^p \otimes T_s^q & \xrightarrow{h} & T_{r+s}^{p+q} \\
 & & & & \downarrow i \\
(T_p^r \otimes T_q^s)^* & \xrightarrow{\quad h^* \quad} & (T_{p+q}^{r+s})^* & \xrightarrow{\phi_{p+q}^{r+s}} & T_{r+s}^{p+q}
\end{array}$$

are isomorphisms it follows that

$$(h^*)^{-1} \circ (\phi_{p+q}^{r+s})^{-1} \circ h \circ (\phi_p^r \otimes \phi_q^s)$$

is an isomorphism of $(T_p^r)^* \otimes (T_q^s)^*$ onto $(T_p^r \otimes T_q^s)^*$.

COROLLARY 2. *The abstract pth tensorial power $\otimes^p T_1$ is isomorphic to T_p under h.*

PROOF. This follows immediately from the theorem.

Symmetric Tensors. Let $S\colon \otimes^p T_1 \to \otimes^p T_1$ be the operator of symmetry defined by

$$S(\omega^1 \otimes \cdots \otimes \omega^p) = \frac{1}{p!} \sum_\sigma \omega^{\sigma(1)} \otimes \cdots \otimes \omega^{\sigma(p)}$$

where σ runs through all permutations of p objects. The *symmetric product* $\vee^p T_1$ is defined to be the F-module $\operatorname{Im} S \subset \otimes^p T_1$. Denote by $i\colon \vee^p T_1 \to \otimes^p T_1$ the inclusion homomorphism. On the other hand consider the submodule $S_p \subset T_p$ of *symmetric tensors*; let $i'\colon S_p \to T_p$

be the inclusion homomorphism. Φ is in S_p if and only if for any permutation σ and any vector fields $\xi_i, \cdots, \xi_p$ we have $(\sigma\Phi)(\xi_1, \cdots, \xi_p) = \Phi(\xi_{\sigma(1)}, \cdots, \xi_{\sigma(p)}) = \Phi(\xi_1, \cdots, \xi_p)$.

COROLLARY 3. *The F-isomorphism $h: \otimes^p T_1 \to T_p$ induces an F-isomorphism $\bar{h}: \vee^p T_1 \to S_p$ such that $h \circ i = i' \circ \bar{h}$.*

PROOF. One easily verifies that $\operatorname{Im}(h \circ i) = \operatorname{Im} i'$; then defines $\bar{h} = i'^{-1} \circ h \circ i$. q.e.d.

Exterior Forms. The pth exterior power $\wedge^p T_1$ is the quotient F-module $\otimes^p T_1/N$ where $N \subset \otimes^p T_1$ is the submodule generated by the elements $\Phi \in \otimes^p T_1$ such that $\tau\Phi = \Phi$ for some transposition τ. The operator $\mathfrak{A}$ of antisymmetrization is defined by

$$\mathfrak{A}\Phi = \frac{1}{p!} \sum_\sigma \epsilon_\sigma \cdot \sigma\Phi,$$

where σ runs over all permutations of p objects and ϵ_σ is the sign of the permutation σ. Let $N_1 \subset \otimes^p T_1$ be the kernel of $\mathfrak{A}$. According to Bourbaki, *Algèbre*, Chapter III, p. 60, 2^e edition, we have $N \subset N_1$. But on the other hand by Proposition 3 in Bourbaki, loc. cit., p. 58, for any $\Phi \in \otimes^p T_1$ and any permutation σ, we have $\Phi - \epsilon_\sigma \cdot \sigma\Phi \in N$. Therefore

$$\sum_\sigma (\Phi - \epsilon_\sigma \cdot \sigma\Phi) \in N$$

or $p!\Phi - \mathfrak{A}\Phi \in N$. Hence, if $\Phi \in N_1$, then $\Phi \in N$. Whence $N_1 = N$.

Now let $h: \otimes^p T_1 \to T_p$ be the isomorphism of our main theorem, $\pi: \otimes^p T_1 \to \wedge^p T_1$ the canonical projection homomorphism and $\mathfrak{A}': T_p \to A_p \subset T_p$ the antisymmetrization map in T_p. A_p consists of the antisymmetric tensors or global p-forms.

COROLLARY 4. *$h: \otimes^p T_1 \to T_p$ induces an F-isomorphism $\bar{h}: \wedge^p T_1 \to A_p$ such that $\bar{h} \circ \pi = \mathfrak{A}' \circ h$.*

PROOF. We have shown that $N_1 = N$ or equivalently $\ker \mathfrak{A} = \ker \pi = N$. It is easy to verify by computation that $h \circ \mathfrak{A} = \mathfrak{A}' \circ h$. Then $k(\ker \pi) = h(\ker \mathfrak{A}) = \ker \mathfrak{A}'$ and this proves that $\bar{h}$ exists and is an isomorphism.

We proceed to the proof of the main theorem.

LEMMA I. *Let U_α be a system of coordinate neighborhoods on M such that*

$$U_\alpha \cap U_\beta = \emptyset \quad \text{if } \alpha \neq \beta$$

and A_α be a compact subset of U_α. Then there exists a system of n tensor fields $\omega^i \in T_1$ such that the n tensors $\omega^i(x)$ are linearly independent for every $x \in \bigcup_\alpha A_\alpha$.

Proof. In each U_α there exists a system of n tensor fields $\tilde{\omega}_\alpha^i$ of order 1 in U_α such that the tensors $\tilde{\omega}_\alpha^i(x)$, $(i=1 \cdots n)$ are linearly independent at every point $x \in U_\alpha$. Now let h_α be a C^k-function on M such that the carrier of h_α is compact and contained in U_α and that

$$h_\alpha = 1 \quad \text{in } A_\alpha.$$

Define ω_α^i by

$$\overset{i}{\omega}_\alpha = \begin{cases} h_\alpha \overset{i}{\tilde{\omega}}_\alpha & \text{in } U_\alpha, \\ 0 & \text{in } M - U_\alpha \end{cases}$$

and ω^i by

$$\omega^i = \sum_\beta \overset{i}{\omega}_\beta.$$

Then ω^i is a tensor field of order 1 on M. Now let $x \in \bigcup_\alpha A_k$ be an arbitrary point. Since the compact sets A_α are mutually disjoint the point x belongs to precisely one of them, say to A_α. This implies that

$$\omega^i(x) = h_\alpha(x) \overset{i}{\tilde{\omega}}_\alpha(x) = \overset{i}{\tilde{\omega}}_\alpha(x)$$

i.e., the tensors $\omega^i(x)$ are linearly independent.

Lemma II. *Consider the sets U_α and A_α given in Lemma I. Then there exists a system of n tensor fields $\omega^i \in T_1$ with the following property: Every tensor field $\phi \in T_p$ whose carrier is contained in $\bigcup_\alpha A_\alpha$ can be written in the form*

$$\phi = \sum_{(\nu)} \lambda_{\nu_1 \cdots \nu_p} \omega^{\nu_1} \cdots \omega^{\nu_p}$$

where the $\lambda_{\nu_1 \cdots \nu_p}$ are scalar functions on M whose carriers are contained in $\bigcup_\alpha A_\alpha$.

Proof. Choose a system of open sets B_α with compact closure such that

$$A_\alpha \subset B_\alpha \subset \overline{B}_\alpha \subset U_\alpha.$$

Applying Lemma I to the compact sets $\overline{B}_\alpha$ we obtain n tensor fields $\omega^i \in T_i$ such that the tensors $\omega^i(x)$ are linearly independent for every

$x \in \bigcup_\alpha B_\alpha$. Hence, we can write

$$(1) \qquad \phi(x) = \sum_{(\nu)} \tilde{\lambda}_{\nu_1 \cdots \nu_p}(x) \omega^{\nu_1}(x) \cdots \omega^{\nu_p}(x), \qquad x \in \bigcup_\alpha B_\alpha,$$

where the coefficients are C^k-functions in $\bigcup_\alpha B_\alpha$. Since the carrier of Φ is contained in $\bigcup_\alpha A_\alpha$ the same must be true for every function $\tilde{\lambda}_{\nu_1 \cdots \nu_p}$. Hence, a system of C^k-functions $\lambda_{\nu_1 \cdots \nu_p}$ can be defined on M by

$$\lambda_{\nu_1 \cdots \nu_p} = \begin{cases} \tilde{\lambda}_{\nu_1 \cdots \nu_p} & \text{in } \bigcup_\alpha B_\alpha, \\ 0 & \text{in } M - \bigcup_\alpha B_\alpha. \end{cases}$$

Then

$$(2) \qquad \Phi(x) = \sum_{(\nu)} \lambda_{\nu_1 \cdots \nu_p}(x) \omega^{\nu_1}(x) \cdots \omega^{\nu_p}(x)$$

for every point $x \in M$. In fact, if $x \in \bigcup_\alpha B_\alpha$, the relation (2) follows from (1) and otherwise both sides of (2) are zero.

LEMMA III. *With U_α and A_α as in Lemma* I *consider any* $2r$ *tensor fields* $\Phi^j \in T_p$ and $\Psi^j \in T_p$ *where the carriers of the* Φ^j *are contained in* $\bigcup_\alpha A_\alpha$. *Then the relation*

$$(3) \qquad \sum_j \Phi^j \cdot \Psi^j = 0$$

implies that

$$\sum_j \Phi^j \otimes \Psi^j = 0.$$

PROOF. Choose the B_α as in Lemma II and let ω^i be the tensor fields constructed in Lemma II. Then Φ^j can be written as

$$\Phi^j = \sum_{(\nu)} \overset{j}{\lambda}_{\nu_1 \cdots \nu_p} \omega^{\nu_1} \cdots \omega^{\nu_p}.$$

It follows from (3) that

$$(4) \qquad \sum_j \sum_{(\nu)} \overset{j}{\lambda}_{\nu_1 \cdots \nu_p} \omega^{\nu_1} \cdots \omega^{\nu_p} \cdot \Psi^j = 0.$$

Since the tensors $\omega^i(x)$ are linearly independent for every $x \in \bigcup_\alpha B_\alpha$ the relation (4) implies that

$$\sum \lambda^{j}_{\nu_1\cdots\nu_p}(x)\Psi^j(x) = 0, \qquad x \in \bigcup_\alpha B_\alpha. \tag{5}$$

by Lemma II the carrier of $\lambda^j_{\nu_1\cdots\nu_p}$ is contained in $\bigcup_\alpha A_\alpha$ and hence in $\bigcup_\alpha B_\alpha$. Thus (5) holds for every $x \in M$, i.e.

$$\sum_j \lambda^{j}_{\nu_1\cdots\nu_p}\Psi^j = 0.$$

Now the bilinearity of the tensor product yields

$$\sum_j \Phi^j \otimes \Psi^j = \sum_j \left(\sum_{(\nu)} \lambda^{j}_{\nu_1\cdots\nu_p}\omega^{\nu_1} \cdots \omega^{\nu_p}\right) \otimes \Psi^j$$
$$= \sum_{(\nu)} \left(\omega^{\nu_1} \cdots \omega^{\nu_p} \otimes \sum_j \lambda^{j}_{\nu_1\cdots\nu_p}\Psi^j\right) = 0.$$

LEMMA IV. *Let M be a paracompact n-dimensional manifold. Then there exists a locally finite covering by open sets V^k_α where $k=0, 1, \cdots, n_0$ ($n_0 \leqq n$) and $\alpha \in \mathfrak{J}_k$ ($\mathfrak{J}_k$ index sets) subject to the following conditions*:
(i) $\overline{V}^k_\alpha$ *is compact*,
(ii) V^k_α *is contained in a coordinate neighborhood*,
(iii) $V^k_\alpha \cap V^k_\beta = \varnothing$ *for* $\alpha \neq \beta$.

PROOF. Since M is a manifold, we may consider the covering $\{U\}$ consisting of all relatively compact coordinate neighborhoods. M is paracompact and hence there is a locally-finite refinement $\{S\}$ of $\{U\}$. As a paracompact space, M is normal; M has dimension n, hence $\{S\}$ has a refinement $\{R_\mu\}$ of order $\leqq n$. (See C. H. Dowker, Amer. J. Math. (1947), p. 211, together with W. Hurewicz, *Dimension theory*, Princeton Univ. Press, Princeton, N. J., 1941; Theorem V8, p. 67.) Again, since M is paracompact, there is a locally finite refinement $\{W_\mu\}$ of $\{R_\mu\}$ with index set a subset of the former index set and $W_\mu \subset R_\mu$. For if $\{Z_\beta\}$ is a locally-finite refinement of $\{R_\mu\}$ choose $\mu(\beta)$ such that $Z_\beta \subset R_{\mu(\beta)}$ and put $W_\mu = \bigcup_{\mu(\beta)=\mu} Z_\beta$. Then $\{W_\mu\}$ is locally finite and of order $n_0 \leqq n$. There exists a partition of unity $\{\phi_\mu\}$ with carrier $\phi_\mu \subset W_\mu$. Of course carrier ϕ_μ is compact. Given $k+1$ different indices $\mu_0, \cdots, \mu_k$ put $\alpha = (\mu_0 \cdots \mu_k)$ and consider the sets

$$V^k_\alpha = \left\{x \mid x \in M, \phi_\mu(x) < \mathrm{Min}\,[\phi_{\mu_0}(x), \cdots, \phi_{\mu_k}(x)]\ \mu \neq \mu_0, \cdots, \mu_k\right\}.$$

Each V^k_α is open and $V^k_\alpha \cap V^k_\beta = \varnothing$ for $\alpha \neq \beta$. Furthermore,

$$V^k_\alpha \subset (\text{carrier } \phi_{\mu_0}) \cap \cdots \cap (\text{carrier } \phi_{\mu_k}).$$

Hence $\overline{V}_\alpha^k$ is compact and contained in some W_μ. Therefore it is contained in a coordinate neighborhood. Since the order of the covering $\{W_\mu\}$ is n_0, for $k>n_0$ the sets V_α^k are void. The sets V_α^k $(0 \leqq k \leqq n_0)$ cover M since for every $x \in M$ some $\phi_\mu(x)>0$. The covering $\{V_\alpha^k\}$ is locally finite since $\{W_\mu\}$ is and hence it has all desired properties.

THEOREM. *The homomorphism h is an isomorphism onto T_{p+q}.*

PROOF. Consider the covering

$$M = \bigcup_{k=0}^{n_0} \bigcup_\alpha V_\alpha^k \tag{6}$$

constructed in Lemma IV. Since M is paracompact and the covering $\{V_\alpha^k\}$ is locally finite we can choose an open subset W_α^k in each V_α^k such that

$$\overline{W}_\alpha^k \subset V_\alpha^k \tag{7}$$

and

$$\bigcup_{k=0}^{n_0} \bigcup_\alpha W_\alpha^k = M. \tag{8}$$

It follows from (7) and the property (i) in Lemma IV that the closures $\overline{W}_\alpha^k$ are compact.

Put

$$W^k = \bigcup_\alpha W_\alpha^k \qquad (k = 0 \cdots n)$$

and let f^k be a partition of unity subordinate to the covering $\{W^k\}$. Given an arbitrary tensor field $\Omega \in T_{p+q}$ consider the tensor fields $\Omega^k = f^k\Omega$. The carrier of Ω^k is contained in W^k. Applying Lemma II with

$$U_\alpha = V_\alpha^k \quad \text{and} \quad A_\alpha = \overline{W}_\alpha^k,$$

we see that Ω^k can be written as

$$\Omega^k = \sum_{(\nu)} \lambda^k_{\nu_1 \cdots \nu_{p+q}} \omega^{\nu_1} \cdots \omega^{\nu_{p+q}} \tag{9}$$

where $\omega^\nu \in T_1$ and $\lambda^k_{\nu_1 \cdots \nu_{p+q}} \in F$. Introducing the tensor fields

$$\Phi^{\nu_1 \cdots \nu_p} = \omega^{\nu_1} \cdots \omega^{\nu_p};$$

$$\Psi^{\nu_{p+1}\cdots\nu_{p+q}} = \omega^{\nu_{p+1}} \cdots \omega^{\nu_{p+q}}$$

we obtain from (9)

$$\begin{aligned}\overset{k}{\Omega} &= \sum_{(\nu)} \overset{k}{\lambda}_{\nu_1\cdots\nu_{p+q}} \Phi^{\nu_1\cdots\nu_p}\Psi^{\nu_{p+1}\cdots\nu_{p+q}} \\ &= h\left(\sum_{(\nu)} \overset{k}{\lambda}_{\nu_1\cdots\nu_{p+q}} \Phi^{\nu_1\cdots\nu_p} \otimes \Psi^{\nu_{p+1}\cdots\nu_{p+q}} \right).\end{aligned}$$

Summation over K yields

$$\Omega = h\left(\sum_{k=0}^{n} \sum_{(\nu)} \overset{k}{\lambda}_{\nu_1\cdots\nu_{p+q}} \Phi^{\nu_1\cdots\nu_p} \otimes \Psi^{\nu_{p+1}\cdots\nu_{p+q}} \right).$$

This relation shows that h is an *onto* map.

To prove that h is one-to-one suppose that

$$h\left(\sum_j \Phi^j \otimes \Psi^j \right) = 0$$

where

$$\Phi^j \in T_p \quad \text{and} \quad \Psi^j \in T_q.$$

Then $\sum_j \Phi\cdot^j\Psi^j = 0$ and multiplication by f^k yields

$$\sum_j f^k \Phi^j \Psi^j = 0.$$

Since the carrier of $f^k\Phi^j$ is contained in $W^k \subset \bigcup_\alpha \overline{W}^k_\alpha$ we can apply Lemma III with

$$U_\alpha = \overset{k}{V}_\alpha \quad \text{and} \quad A_\alpha = \overline{\overset{k}{W}}_\alpha.$$

We thus obtain

$$\sum_j f^k \Phi^j \otimes \Psi^j = 0$$

and summing over k

$$\sum_j \Phi^j \otimes \Psi^j = 0.$$

The above theorem is thereby proved.

UNIVERSITY OF TORONTO