SURFACES OF CONSTANT MEAN CURVATURE

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1. **Introduction.** Let S be a surface immersed in euclidean space $\boldsymbol{R}^3$ with constant mean curvature H. In a recent note [**3**] we proved that the quadratic differential form $-HI+II$ is a flat Lorentz metric on the complement of the umbilic set of S. Here the result is used to set up a certain type of isothermal local coordinate system on S. The main consequences are:

(i) an obstruction theory, which tells one when an isometry of connected surfaces of the same constant mean curvature is a congruence;[2]

(ii) Gauss curvature on S is set up as a solution to a nonlinear elliptic boundary value problem; and

(iii) construction of local surfaces of any given constant mean curvature.

2. **Notation.** S denotes a surface with a fixed immersion $\nu: S \to \boldsymbol{R}^3$. If ξ is a smooth choice of unit normal defined over an open set $U \subset S$, then we recall the fundamental forms of the immersion:

$I = d\nu \cdot d\nu$, *first fundamental form*;

$II = d\nu \cdot d\xi$, *second fundamental form*;

$III = d\xi \cdot d\xi$, *third fundamental form*.

$I = d\nu^2$ is the riemannian metric induced on S by the immersion. The eigenvalues of II relative to I are the *principle curvatures*, denoted k_i. As usual we have functions H, K on S given by

$$H = \tfrac{1}{2}\{k_1 + k_2\}, \text{ mean curvature};$$

$$K = k_1 k_2, \text{ Gauss curvature}.$$

They define the quadratic differential form

$$\Omega = -HI + II, \text{ modified fundamental form}.$$

The eigenvalues of Ω relative to I are $k_i - \frac{1}{2}(k_1+k_2) = \pm\frac{1}{2}(k_1 - k_2)$. Thus Ω is a pseudo-riemannian metric of Lorentz signature (Lorentz metric) on the open subset

$$S_\Omega = \{x \in S: k_1(x) \neq k_2(x)\}$$

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[2] In other words, when the isometry is the restriction of a rigid motion of the ambient euclidean space $\boldsymbol{R}^3$.

of S. We view S_Ω as a Lorentz surface with metric Ω. Recall that a point $x \in S$ is called *umbilic* if $k_1(x) = k_2(x)$; thus S_Ω is the complement of the umbilic set of S.

3. **Special coordinates on** S_Ω. The results of this note are based on the following observation.

3.1. THEOREM. *Let S be a surface immersed in $\mathbf{R}^3$ with constant mean curvature H. Let K denote Gauss curvature and define*[3] *a function*

$$\lambda = -\tfrac{1}{2}\log(H^2 - K) \text{ on } S_\Omega. \tag{3.2}$$

If $x \in S_\Omega$, then x has a local coordinate neighborhood[4] (U, u) *with $U \subset S_\Omega$ and*

$$I = e^\lambda\{du^1 \otimes du^1 + du^2 \otimes du^2\}; \tag{3.3}$$

$$II = (He^\lambda + 1)du^1 \otimes du^1 + (He^\lambda - 1)du^2 \otimes du^2; \tag{3.4}$$

$$k_1 = H + e^{-\lambda}, \quad k_2 = H - e^{-\lambda}, \quad K = H^2 - e^{-2\lambda}. \tag{3.5}$$

If (V, v) is another local coordinate neighborhood of x with these properties, then $v^i = \pm u^i + c^i$, c^i constant, on each component of $U \cap V$.

PROOF. Let the principle curvature be numbered so that $k_1 > k_2$ on S_Ω. Given $x \in S_\Omega$ we choose a neighborhood $W \subset S_\Omega$ of x which carries an $\boldsymbol{I}$-orthonormal moving frame $\{X_1, X_2\}$ such that X_i is a principle vector with principle curvature k_i. We have seen [**3**, Corollary 4.11] that the connection form of the Lorentz surface S_Ω is identically zero in the Ω-orthonormal moving frame $\{Y_1, Y_2\}$, where $Y_i = \{\frac{1}{2}(k_1 - k_2)\}^{1/2} X_i$. It follows that x has a local coordinate neighborhood (U, u) such that $U \subset W$ and $\partial/\partial u^i = Y_i$. Now

$$\Omega = du^1 \otimes du^1 - du^2 \otimes du^2 \text{ in } U.$$

On the other hand, $\boldsymbol{I}$ and $\boldsymbol{II}$ are diagonalized by $\{X_1, X_2\}$, hence also by $\{Y_1, Y_2\} = \{\partial/\partial u^1, \partial/\partial u^2\}$. Thus

$$I = \sum_1^2 g_i du^i \otimes du^i \quad \text{and} \quad II = \sum_1^2 b_i du^i \otimes du^i$$

in U. This tells us

$$b_i = k_i g_i, \quad -Hg_1 + b_1 = 1, \quad -Hg_2 + b_2 = -1.$$

We compute

[3] Here we must observe that $H^2 - K > 0$ on S_Ω; for $H^2 - K = \frac{1}{4}(k_1 - k_2)^2$.

[4] U is the neighborhood and $u = (u^1, u^2)$ is the local coordinate.

$$2H = k_1 + k_2 = \frac{b_1}{g_1} + \frac{b_2}{g_2} = \frac{Hg_1 + 1}{g_1} + \frac{Hg_2 - 1}{g_2} = 2H + \frac{1}{g_1} - \frac{1}{g_2} \cdot$$

Thus $g_1 = g_2$, which must be positive because I is positive definite. Now $g_1 = g_2 = e^\lambda$ for some function λ defined on U. We compute

$$b_1 = Hg_1 + 1 = He^\lambda + 1, \qquad k_1 = b_1/g_1 = H + e^{-\lambda};$$
$$b_2 = Hg_2 - 1 = He^\lambda - 1, \qquad k_2 = b_2/g_2 = H - e^{-\lambda};$$
$$K = k_1 k_2 = H^2 - e^{-2\lambda}, \quad \text{so} \quad \lambda = -\tfrac{1}{2}\log(H^2 - K).$$

This proves (3.3), (3.4) and (3.5).

For the uniqueness, observe that $\{\partial/\partial v^1, \partial/\partial v^2\}$ diagonalizes I and II with first coefficient greater than second in II. Thus $\partial/\partial v^i$ is a principle vector with principle curvature k_i on S. As $\Omega(\partial/\partial v^i, \partial/\partial v^i) = \Omega(\partial/\partial u^i, \partial/\partial u^i) = \pm 1 \neq 0$, now $\partial/\partial v^i = \pm \partial/\partial u^i$, so $dv^i = \pm du^i$. q.e.d.

4. **The Mainardi-Codazzi equations.** Let (U, u) be a connected local coordinate neighborhood on a surface S immersed in $\boldsymbol{R}^3$. Suppose that the fundamental forms are given by

$$(4.1) \quad I = e^\lambda\{du^1 \otimes du^1 + du^2 \otimes du^2\} \text{ and } II = \textstyle\sum_{ij} b_{ij} du^i \otimes du^j.$$

Then the Christoffel symbols are easily computed:

$$(4.2) \quad \overset{1}{\Gamma}_{11} = -\overset{1}{\Gamma}_{22} = \overset{2}{\Gamma}_{12} = \frac{1}{2}\frac{\partial\lambda}{\partial u^1}; \quad \overset{1}{\Gamma}_{12} = -\overset{2}{\Gamma}_{11} = \overset{2}{\Gamma}_{22} = \frac{1}{2}\frac{\partial\lambda}{\partial u^2} \cdot$$

Thus the Mainardi-Codazzi equations reduce to

$$(4.3) \quad \begin{aligned} &\frac{\partial b_{11}}{\partial u^2} - \frac{\partial b_{12}}{\partial u^1} = \frac{1}{2}(b_{11} + b_{22})\frac{\partial\lambda}{\partial u^2} \quad \text{and} \\ &\frac{\partial b_{22}}{\partial u^1} - \frac{\partial b_{12}}{\partial u^2} = \frac{1}{2}(b_{11} + b_{12})\frac{\partial\lambda}{\partial u^1} \cdot \end{aligned}$$

Now suppose that our surface S has constant mean curvature H. Let $z = u^1 + (-1)^{1/2}u^2$, complex local coordinate, and define

$$\phi(z) = (b_{11} - b_{22}) + 2(-1)^{1/2} b_{12}.$$

As $2H = b_{11}e^{-\lambda} + b_{22}e^{-\lambda} = (b_{11} + b_{22})e^{-\lambda}$ is constant, (4.3) says that $\partial/\partial\bar{z} = \frac{1}{2}\{\partial/\partial u^1 + (-1)^{1/2}\partial/\partial u^2\}$ annihilates ϕ; thus ϕ is a holomorphic function of z. Let f be the function on U defined by

$$b_{11} = He^\lambda + f, \qquad b_{22} = He^\lambda - f.$$

Suppose that Gauss curvature satisfies

$$K = H^2 - e^{-2\lambda}, \quad \text{i.e.,} \quad \lambda = -\tfrac{1}{2}\log(H^2 - K).$$

Then

$$H^2 e^{2\lambda} - 1 = Ke^{2\lambda} = b_{11}b_{22} - b_{12}^2 = H^2 e^{2\lambda} - (f^2 + b_{12}^2),$$

so $f^2+b_{12}^2=1$. But $\phi=2(f+(-1)^{1/2}b_{12})$ is holomorphic; now the maximum modulus principle says that ϕ is constant; thus f and b_{12} are constant.

Notice that $U\subset S_\Omega$ by the assumption $H^2-K=e^{-2\lambda}>0$. Cutting U down if necessary, Theorem 3.1 gives us a local coordinate v on U in which $\boldsymbol{I} = e^\lambda\{dv^1 \otimes dv^1 + dv^2 \otimes dv^2\}$ and $\boldsymbol{II} = (He^\lambda + 1)dv^1 \otimes dv^1 +(He^\lambda-1)dv^2\otimes dv^2$. If α is the oriented angle from $\partial/\partial u^1$ to $\partial/\partial v^1$, the two expressions for $\boldsymbol{I}$ give

$$dv^1 = \cos\alpha\, du^1 + \sin\alpha\, du^2 \quad \text{and} \quad dv^2 = -\sin\alpha\, du^1 + \cos\alpha\, du^2.$$

Equating coefficients of $du^1\otimes du^1$ in the two expressions for $\boldsymbol{II}$,

$$\begin{aligned} He^\lambda + f = b_{11} &= (He^\lambda + 1)\cos^2\alpha + (He^\lambda - 1)\sin^2\alpha \\ &= He^\lambda + \{\cos^2\alpha - \sin^2\alpha\}. \end{aligned}$$

Thus $f=\cos^2\alpha-\sin^2\alpha=\cos(2\alpha)$. Similarly $b_{12}=2\cos\alpha\sin\alpha=\sin(2\alpha)$. Now α is constant, and $v^1=\cos\alpha u^1+\sin\alpha u^2+c^1$ and $v^2=-\sin\alpha u^1+\cos\alpha u^2+c^2$ for some constants c^i. We summarize as follows.

4.4. THEOREM. *Let S be a surface immersed in $\boldsymbol{R}^3$ with constant mean curvature H, and define $\lambda=-\frac{1}{2}\log(H^2-K)$ on S_Ω. Let (U, u) be a connected local coordinate neighborhood such that $U\subset S_\Omega$ and $\boldsymbol{I}=e^\lambda\{du^1\otimes du^1+du^2\otimes du^2\}$. Then there is a constant α such that*

$$\text{(4.5)} \qquad \begin{aligned} \boldsymbol{II} = {} & (He^\lambda + \cos 2\alpha)du^1 \otimes du^1 + 2\sin 2\alpha\, du^1du^2 \\ & + (He^\lambda - \cos 2\alpha)du^2 \otimes du^2. \end{aligned}$$

Let c^i be constants, $v^1=\cos\alpha u^1+\sin\alpha u^2+c^1$ and $v^2=-\sin\alpha u^1+\cos\alpha u^2+c^2$. Then $v=(v^1, v^2)$ is a local coordinate on U, α is the angle from $\partial/\partial u^1$ to $\partial/\partial v^1$, and

$$\text{(4.6)} \qquad \begin{aligned} \boldsymbol{I} &= e^\lambda\{dv^1 \otimes dv^1 + dv^2 \otimes dv^2\} \quad \textit{and} \\ \boldsymbol{II} &= (He^\lambda + 1)dv^1 \otimes dv^1 + (He^\lambda - 1)\, dv^2 \otimes dv^2. \end{aligned}$$

5. **Obstruction to a congruence.** The following result generalizes the fact that an isometry of small patches of a right circular cylinder is a congruence only when it preserves the direction of the axis of the cylinder.

5.1. THEOREM. *Let S, S' and S'' be connected surfaces embedded in*

$\boldsymbol{R}^3$ with the same constant mean curvature H, which are not open subsets of a plane or a sphere. Then to every isometry $f\colon S\to S'$ we have a real number $\alpha(f)$, defined up to addition of an integral multiple of π, specified by the property: if $x\in S_\Omega$ and (U, u) is a local coordinate neighborhood of x given by Theorem 3.1 *then $f^*\boldsymbol{II}' = \{He^\lambda + \cos 2\alpha(f)\} du^1\otimes du^1 + 2\sin 2\alpha(f) du^1du^2 + \{He^\lambda - \cos 2\alpha(f)\} du^2\otimes du^2$ in U.*

α has the properties:

(i) *f extends to a rigid motion of $\boldsymbol{R}^3$, if and only if $\alpha(f)\equiv 0 \bmod \pi$;*

(ii) $\alpha(f^{-1}) = -\alpha(f)$;

(iii) *if $g\colon S'\to S''$ is an isometry, then $\alpha(g\cdot f) = \alpha(f) + \alpha(g)$;*

(iv) *two isometries $f, g\colon S\to S'$ differ by a rigid motion of $\boldsymbol{R}^3$, if and only if $\alpha(f)\equiv\alpha(g) \bmod \pi$.*

Given $x\in S_\Omega$ and a real number b, there is a neighborhood V of x, a surface W of constant mean curvature H, and an isometry $h\colon V\to W$, such that $\alpha(h) = b$.

The proof is based on the standard fact [2]:

5.2. Lemma. *Let S be a connected surface immersed in $\boldsymbol{R}^3$ with constant mean curvature. If some point of S is not umbilic, then the umbilics of S are isolated.*

{The function ϕ of §4 is holomorphic, and the umbilics of S in the domain of ϕ are just the zeroes of ϕ.}

Now the surfaces S, S' and S'' have all umbilics isolated; for an all-umbilic surface is an open subset of a plane or a sphere. Thus S_Ω (resp. S'_Ω, resp. S''_Ω) is arcwise connected and dense in S (resp. S', resp. S''). Let K, K' and K'' denote their Gauss curvature functions and let $f\colon S\to S'$ be an isometry. Then $f(S_\Omega) = S'_\Omega$ because $K'(f(x)) = K(x)$, $S_\Omega = \{x\in S\colon H^2 - K(x) > 0\}$ and $S'_\Omega = \{f(x)\in S'\colon H^2 - K'(f(x)) > 0\}$. Similarly, the functions $\lambda = -\frac{1}{2}\log(H^2 - K)$ on S_Ω and $\lambda' = -\frac{1}{2}\log(H^2 - K')$ on S'_Ω are related by $\lambda = \lambda'\cdot f$.

Let $x\in S_\Omega$ and choose a connected local coordinate neighborhood (U, u) according to Theorem 3.1. Then $\boldsymbol{I} = e^\lambda\{du^1\otimes du^1 + du^2\otimes du^2\}$ and $\boldsymbol{II} = (He^\lambda + 1)du^1\otimes du^1 + (He^\lambda - 1)du^2\otimes du^2$. Let $W = f(U)$, $w^i(f(z)) = u^i(z)$ for $z\in U$; then (W, w) is a connected local coordinate neighborhood of $f(x)$. $U\subset S_\Omega$ implies $W\subset S'_{\Omega_2}\cdot f^*\boldsymbol{I}' = \boldsymbol{I}$ because f is an isometry, so $f^*\boldsymbol{I}' = e^\lambda\{du^1\otimes du^1 + du^2\otimes du^2\} = e^{\lambda'\cdot f}\{d(w^1\cdot f)\otimes d(w^1\cdot f) + d(w^2\cdot f)\otimes d(w^2\cdot f)\}$; thus $\boldsymbol{I}' = e^{\lambda'}\{dw^1\otimes dw^1 + dw^2\otimes dw^2\}$. Applying Theorem 4.4 to S', we have a number $\alpha_f(x)$ such that $\boldsymbol{II}' = He^{\lambda'} + \cos 2\alpha_f(x))dw^1\otimes dw^1 + 2\sin 2\alpha_f(x)dw^1dw^2 + (He^{\lambda'} - \cos 2\alpha_f(x))dw^2\otimes dw^2$. Thus $f^*\boldsymbol{II}' = (He^\lambda + \cos 2\alpha_f(x))du^1\otimes du^1 + 2\sin 2\alpha_f(x)du^1du^2 + (He^\lambda - \cos 2\alpha_f(x))du^2\otimes du^2$. This specifies $\alpha_f(x)$ up to an integral multiple of π. The uniqueness part of Theorem 3.1 says that $\alpha_f(x)$ is well defined up to an integral multiple of π.

Let C be the circle which is the real numbers modulo π. We have a map $\alpha_f: S_\Omega \to C$. If $x \in S_\Omega$ then α_f is constant on a neighborhood of x. As S_Ω is connected, now α_f is constant. Let $\alpha(f)$ denote its value. We have proved the existence of a number $\alpha(f)$ defined modulo π and specified by $f^*\mathbf{II}'$ as required.

If $\alpha(f) \equiv 0 \bmod \pi$ if and only if $\cos 2\alpha(f) = 1$ and $\sin 2\alpha(f) = 0$, which is equivalent to $f^*\mathbf{II}' = \mathbf{II}$. In that case $f: S \to S'$ is a diffeomorphism of connected surfaces in $\mathbf{R}^3$ such that $f^*\mathbf{I}' = \mathbf{I}$ and $f^*\mathbf{II}' = \mathbf{II}$, and a classical theorem says that f extends to a rigid motion of $\mathbf{R}^3$. This proves (i).

For (iii) we have a local coordinate v on $W \subset S'$ given by $(\alpha_f = \alpha(f))$ $v^1 = \cos(\alpha_f)w^1 + \sin(\alpha_f)w^2$ and $v^2 = -\sin(\alpha_f)w^1 + \cos(\alpha_f)w^2$, and $\mathbf{II}' = (He^{\lambda'} + 1)dv^1 \otimes dv^1 + (He^{\lambda'} - 1)dv^2 \otimes dv^2$ on W. Let $X = g(W) \subset S''_\Omega$ and let x be the local coordinate on X with $x \cdot g = v$. Define y on X by $y^1 = \cos(\alpha_g)x^1 + \sin(\alpha_g)x^2$ and $y^2 = -\sin(\alpha_g)x^1 + \cos(\alpha_g)x^2$; then $\mathbf{II}'' = (He^{\lambda''} + 1)dy^1 \otimes dy^1 + (He^{\lambda''} - 1)dy^2 \otimes dy^2$ on X. We compute

$$u^1 = \cos(\alpha_f + \alpha_g)(y^1 \cdot g \cdot f) + \sin(\alpha_f + \alpha_g)(y^2 \cdot g \cdot f).$$
$$u^2 = -\sin(\alpha_f + \alpha_g)(y^1 \cdot g \cdot f) + \cos(\alpha_f + \alpha_g)(y^2 \cdot g \cdot f).$$

Thus $\alpha_{g \cdot f} = \alpha_g + \alpha_f$.

Now (ii) and (iv) are immediate.

Let $x \in S_\Omega$ and $b \in \mathbf{R}$. Choose a local coordinate neighborhood (U, u) of x as in Theorem 3.1. $D = u(U) \subset \mathbf{R}^2$ is the parameter domain. We define functions $g_{11}(u(z)) = g_{22}(u(z)) = e^{\lambda(z)}$, $g_{12} = g_{21} = 0$, $b_{11}(u(z)) = He^{\lambda(z)} + \cos(2b)$, $b_{22}(u(z)) = He^{\lambda(z)} - \cos(2b)$ and $b_{12}(u(z)) = b_{21}(u(z)) = \sin(2b)$. Then the forms

$$\mathbf{I}_0 = \sum g_{ij}(u)du^i \otimes du^j \quad \text{and} \quad \mathbf{II}_0 = \sum b_{ij}(u)du^i \otimes du^j$$

satisfy the Mainardi-Codazzi equations (4.3). As S satisfies the Gauss equation *a priori*, and as $\mathbf{I}_0 = \mathbf{I}$ and $\det(b_{ij}) = H^2e^{2\lambda} - 1$ as for S, now $\mathbf{I}_0$ and $\mathbf{II}_0$ satisfy the Gauss equation. Thus Bonnet's existence theorem says that every $m \in D$ has a neighborhood $V(m) \subset D$ which is parameter domain for a local surface $W \subset \mathbf{R}^3$ with first and second fundamental forms $\mathbf{I}_0$ and $\mathbf{II}_0$. Let w be the coordinate on W, $m = u(x)$, $V = u^{-1}(V(m))$. Then $h = u^{-1} \cdot u|_V$ is a diffeomorphism of V onto W such that $h^*\mathbf{I}_0 = \mathbf{I}_0 = \mathbf{I}$ (so h is an isometry) and $h^*\mathbf{II}_0 = \mathbf{II}_0$ (so $\alpha(h) = b$). q.e.d.

Theorem 5.1 requires S_Ω to be nonempty. Thus we remark:

5.3. Complement to Theorem 5.1. *Let S and S' be connected all-*

umbilic surfaces in $\boldsymbol{R}^3$. *Then any isometry* $f\colon S \to S'$ *extends to a rigid motion of* $\boldsymbol{R}^3$.

For f is a diffeomorphism with $f^*I' = I$, and we need only check that $f^*II' = II$. An all-umbilic surface is an open subset of a plane or a sphere. In the first case $II = 0$ and $K = 0$. In the second case $II = (1/r)I$ and $K = 1/r^2$ where r is the radius of the sphere. As f preserves Gauss curvature, now $f^*II' = II$. q.e.d.

6. **The Gauss Equation.** The Gauss equation for a surface S with first fundamental form $e^\lambda\{du^1 \otimes du^1 + du^2 \otimes du^2\}$ says that Gauss curvature is given by[5]

$$K = -\frac{1}{2} e^{-\lambda} \Delta\lambda, \qquad \Delta = \frac{\partial^2}{\partial u^1 \partial u^1} + \frac{\partial^2}{\partial u^2 \partial u^2}\,. \tag{6.1}$$

Now suppose that S has constant mean curvature H and that our local coordinate neighborhood (U, u) is given by Theorem 3.1. Then $K = H^2 - e^{-2\lambda}$, so (6.1) becomes a nonlinear elliptic equation

$$\Delta\lambda = 2(e^{-\lambda} - H^2 e^{\lambda}). \tag{6.2}$$

We view this as an equation[6] for $K = H^2 - e^{-2\lambda}$.

We regard (6.2) as a boundary value problem on a disc D of radius $r > 0$ in $\boldsymbol{R}^2$. Let b be a continuous function on the boundary ∂D of the disc; we look for a solution $\lambda(u^1, u^2)$ to (6.2) on D, continuous on the closure and with values b on ∂D. Let $h(u^1, u^2)$ be the harmonic function on D with boundary values b. Then we write (6.2) in the form (this defines F)

$$\Delta\eta = 2(e^{-(\eta+h)} - H^2 e^{\eta+h}) \equiv F(\eta), \qquad \eta = \lambda - h, \tag{6.3}$$

and we want a solution η on D vanishing on ∂D.

Following Courant-Hilbert ([**1**, Appendix to Chapter 4]), such solutions η exist provided that certain bounds c, m satisfy $(r + r^2)cm \leqq 1/4$. Here c and m are defined as follows. Let $C^2(D)$ denote the set of all continuous functions on the closure of D which are twice con-

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[5] We remark that this has an interesting expression in complex notation. There $z = u^1 + (-1)^{1/2} u^2$ is the variable, so $dz = du^1 + (-1)^{1/2} du^2$ and $d\bar{z} = du^1 - (-1)^{1/2} du^2$, and the vector fields dual to these forms are $\partial/\partial z = (1/2)\{\partial/\partial u^1 - (-1)^{1/2}\partial/\partial u^2\}$ and $\partial/\partial \bar{z} = (1/2)\{\partial/\partial u^1 + (-1)^{1/2}\partial/\partial u^2\}$. The exterior derivative $d = d' + d''$ where by definition $d'(f) = (\partial f/\partial z)dz$ and $d''(f) = (\partial f/\partial \bar{z})d\bar{z}$ on functions. In particular $d'd''f = \frac{1}{4}\Delta f\, dz \wedge d\bar{z}$. The element of area on S is given by $dA = e^\lambda du^1 \wedge du^2 = (1/2)(-1)^{1/2} e^\lambda \cdot dz \wedge d\bar{z}$. Thus (6.1) can be written in coordinate free form $K dA = -(-1)^{1/2} d'd''\lambda$.

[6] Writing it out in terms of K, one obtains $\Delta K = \{(\partial K/\partial u^1)^2 + (\partial K/\partial u^2)^2\}(H^2 - K) + 4(H^2 - K)^{3/2} - 4H^2(H^2 - K)^{1/2}$, $K < H^2$, which is more difficult to study than is (6.2).

tinuously differentiable in D. Then c is specified by

$$\max\left|\frac{\partial f}{\partial u^i}\right| \leqq cr \text{ l.u.b. } |\Delta f| \quad \text{for } f \in C^2(D),$$

and c is independent of choice of r or D. Define the norm

$$\|f\| = \max|f| + \sum \max\left|\frac{\partial f}{\partial u^i}\right|$$

on $C^2(D)$. Then m is any common bound for all $|F(f)|$ and all $|dF/df|$ with $\|f\| \leqq 1$. A glance at the form of F in (6.3) shows now that m is any common bound for $2(e^{-1-h}+H^2e^{1+h})$ and $2(e^{1-h}+H^2e^{h-1})$. As h achieves its maximum on ∂D, now we may take

$$(6.4)\quad m = \max\left\{2(e^{-1-\beta} + H^2e^{1+\beta}),\ \ 2(e^{1-\beta} + H^2e^{\beta-1})\right\},\quad \beta = \max|b|.$$

In summary, and using the fact that solutions to elliptic equations are analytic,

6.5. THEOREM. *Let D be a disc of radius $r>0$ in $\mathbf{R}^2$, let b be a continuous function on ∂D, and let $\beta=\max_{\partial D}|b|$. If*

$$(6.6)\qquad (r + r^2)^{-1} \geqq 8c\cdot\max\left\{e^{-1-\beta} + H^2e^{1+\beta},\ e^{1-\beta} + H^2e^{\beta-1}\right\},$$

then there exists a continuous function λ on the closure of D, real analytic on D, which satisfies (6.2) *and has values b on ∂D.*

The usual uniqueness condition for an equation $\Delta f=A(f, u)$ for given boundary values is $\partial A/\partial f \geqq 0$. But in our case (6.2) we always have $\partial A/\partial f<0$.

We can now describe the construction of umbilic-free local surfaces of constant mean curvature. Such a surface is specified up to congruence by its first and second fundamental forms, and the condition for two candidates

$$\boldsymbol{I} = \sum g_{ij}du^i \otimes du^j \quad \text{and} \quad \boldsymbol{II} = \sum b_{ij}du^i \otimes du^j$$

to give a surface, is that the g_{ij} and b_{ij} satisfy the Mainardi-Codazzi equations and the Gauss equation.

If H is the mean curvature of the desired surfaces, then Theorem 3.1 allows us to take $\boldsymbol{I}$ and $\boldsymbol{II}$ in the forms

$$\begin{aligned} \boldsymbol{I} &= e^{\lambda}\left\{du^1 \otimes du^1 + du^2 \otimes du^2\right\} \quad \text{and} \\ \boldsymbol{II}_t &= \left\{He^{\lambda} + \cos(2t)\right\}du^1 \otimes du^1 + 2\sin(2t)du^1du^2 \\ &\quad + \left\{He^{\lambda} - \cos(2t)\right\}du^2 \otimes du^2 \end{aligned}$$

for $t=0$. For any constant t, the Mainardi-Codazzi equations (4.3) are satisfied for I and II_t, and the Gauss equation (6.2) is

$$\Delta\lambda = 2(e^{-\lambda} - H^2 e^{\lambda}), \qquad K = H^2 - e^{-2\lambda}.$$

Theorem 6.5 gives the existence of many solutions. Given a local solution λ, there corresponds a well defined congruence class of local surfaces $S_{\lambda,t}$ with I as specified and $II = II_t$; Theorem 5.1 shows that the natural map[7] $f_{s,t}\colon S_{\lambda,s} \to S_{\lambda,t}$ is an isometry and $\alpha(f_{s,t}) = t - s$.

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[7] $f_{s,t}$ is given by $u^i(f_{s,t}(x)) = u^i(x)$.