1. **Introduction.** Let \( S \) be a surface immersed in euclidean space \( \mathbb{R}^3 \) with constant mean curvature \( H \). In a recent note [3] we proved that the quadratic differential form \(-HI + II\) is a flat Lorentz metric on the complement of the umbilic set of \( S \). Here the result is used to set up a certain type of isothermal local coordinate system on \( S \). The main consequences are:

(i) an obstruction theory, which tells one when an isometry of connected surfaces of the same constant mean curvature is a congruence; \(^2\)
(ii) Gauss curvature on \( S \) is set up as a solution to a nonlinear elliptic boundary value problem; and
(iii) construction of local surfaces of any given constant mean curvature.

2. **Notation.** \( S \) denotes a surface with a fixed immersion \( \nu: S \rightarrow \mathbb{R}^3 \). If \( \xi \) is a smooth choice of unit normal defined over an open set \( U \subset S \), then we recall the fundamental forms of the immersion:

\[
I = dv \cdot dv, \text{ first fundamental form}; \\
II = dv \cdot d\xi, \text{ second fundamental form}; \\
III = d\xi \cdot d\xi, \text{ third fundamental form}.
\]

\( I = dv^2 \) is the riemannian metric induced on \( S \) by the immersion. The eigenvalues of \( II \) relative to \( I \) are the principle curvatures, denoted \( k_i \). As usual we have functions \( H, K \) on \( S \) given by

\[
H = \frac{1}{2} \{k_1 + k_2\}, \text{ mean curvature}; \\
K = k_1 k_2, \text{ Gauss curvature}.
\]

They define the quadratic differential form

\[
\Omega = -HI + II, \text{ modified fundamental form}.
\]

The eigenvalues of \( \Omega \) relative to \( I \) are \( k_i - \frac{1}{2}(k_1 + k_2) = \pm \frac{1}{2}(k_1 - k_2) \). Thus \( \Omega \) is a pseudo-riemannian metric of Lorentz signature (Lorentz metric) on the open subset

\[
S_\Omega = \{ x \in S : k_1(x) \neq k_2(x) \}
\]

\(^2\) In other words, when the isometry is the restriction of a rigid motion of the ambient euclidean space \( \mathbb{R}^3 \).

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of \( S \). We view \( S_0 \) as a Lorentz surface with metric \( \Omega \). Recall that a point \( x \in S \) is called umbilic if \( k_1(x) = k_2(x) \); thus \( S_0 \) is the complement of the umbilic set of \( S \).

3. Special coordinates on \( S_0 \). The results of this note are based on the following observation.

3.1. Theorem. Let \( S \) be a surface immersed in \( \mathbb{R}^3 \) with constant mean curvature \( H \). Let \( K \) denote Gauss curvature and define a function

\[
\lambda = -\frac{1}{2} \log(H^2 - K) \text{ on } S_0.
\]

If \( x \in S_0 \), then \( x \) has a local coordinate neighborhood \((U, u)\) with \( U \subset S_0 \) and

\[
I = e^{\lambda} \left( du^1 \otimes du^1 + du^2 \otimes du^2 \right);
\]

\[
II = (He^\lambda + 1)du^1 \otimes du^1 + (He^\lambda - 1)du^2 \otimes du^2;
\]

\[
k_1 = H + e^{-\lambda}, \quad k_2 = H - e^{-\lambda}, \quad K = H^2 - e^{-2\lambda}.
\]

If \((V, v)\) is another local coordinate neighborhood of \( x \) with these properties, then \( v^i = \pm u^i + c^i \), \( c^i \) constant, on each component of \( U \cap V \).

Proof. Let the principle curvature be numbered so that \( k_1 > k_2 \) on \( S_0 \). Given \( x \in S_0 \) we choose a neighborhood \( W \subset S_0 \) of \( x \) which carries an \( I \)-orthonormal moving frame \( \{X_1, X_2\} \) such that \( X_i \) is a principle vector with principle curvature \( k_i \). We have seen [3, Corollary 4.11] that the connection form of the Lorentz surface \( S_0 \) is identically zero in the \( \Omega \)-orthonormal moving frame \( \{Y_1, Y_2\} \), where

\[
Y_i = \left\{ \frac{1}{2}(k_1 - k_2) \right\}^{1/2} X_i.
\]

It follows that \( x \) has a local coordinate neighborhood \((U, u)\) such that \( U \subset W \) and \( \partial/\partial u^i = Y_i \). Now

\[
\Omega = du^1 \otimes du^1 - du^2 \otimes du^2 \text{ in } U.
\]

On the other hand, \( I \) and \( II \) are diagonalized by \( \{X_1, X_2\} \), hence also by \( \{Y_1, Y_2\} = \{\partial/\partial u^1, \partial/\partial u^2\} \). Thus

\[
I = \sum_{i=1}^{2} g_{i} du^i \otimes du^i \quad \text{and} \quad II = \sum_{i=1}^{2} b_i du^i \otimes du^i
\]

in \( U \). This tells us

\[
b_i = k_i g_i, \quad -Hg_1 + b_1 = 1, \quad -Hg_2 + b_2 = -1.
\]

We compute

\[3\] Here we must observe that \( H^2 - K > 0 \) on \( S_0 \); for \( H^2 - K = \frac{1}{2}(k_1 - k_2)^2 \).

\[4\] \( U \) is the neighborhood and \( u = (u^1, u^2) \) is the local coordinate.
\[ 2H = k_1 + k_2 = \frac{b_1}{g_1} + \frac{b_2}{g_2} = \frac{Hg_1 + 1}{g_1} + \frac{Hg_2 - 1}{g_2} = 2H + \frac{1}{g_1} - \frac{1}{g_2}. \]

Thus \( g_1 = g_2 \), which must be positive because \( I \) is positive definite.
Now \( g_1 = g_2 = e^\lambda \) for some function \( \lambda \) defined on \( U \). We compute
\[
\begin{align*}
\quad b_1 &= Ho_1 + 1 = H e^\lambda + 1, \quad k_1 = \frac{b_1}{g_1} = H + e^{-\lambda}; \\
\quad b_2 &= Ho_2 - 1 = H e^\lambda - 1, \quad k_2 = \frac{b_2}{g_2} = H - e^{-\lambda}; \\
K &= k_1 k_2 = H^2 - e^{-2\lambda}, \quad \text{so} \quad \lambda = -\frac{1}{2} \log(H^2 - K).
\end{align*}
\]

This proves (3.3), (3.4) and (3.5).

For the uniqueness, observe that \( \{\partial/\partial v^1, \partial/\partial v^2\} \) diagonalizes \( I \) and \( II \) with first coefficient greater than second in \( II \). Thus \( \partial/\partial v^i \) is a principle vector with principle curvature \( k_i \) on \( S \). As \( \Omega(\partial/\partial v^i, \partial/\partial u^i) = \Omega(\partial/\partial u^i, \partial/\partial u^i) = \pm 1 \neq 0 \), now \( \partial/\partial v^i = \pm \partial/\partial u^i \), so \( dv^i = \pm du^i \). q.e.d.

4. The Mainardi-Codazzi equations. Let \((U, u)\) be a connected local coordinate neighborhood on a surface \( S \) immersed in \( \mathbb{R}^3 \). Suppose that the fundamental forms are given by
\[
\begin{align*}
(I) &= e^\chi \{du^1 \otimes du^1 + du^2 \otimes du^2\} \quad \text{and} \quad (II) = \sum_i b_{ij} du^i \otimes du^i.
\end{align*}
\]
Then the Christoffel symbols are easily computed:
\[
\begin{align*}
\Gamma_{11}^1 &= -\Gamma_{22}^1 = \Gamma_{12}^2 = \frac{1}{2} \frac{\partial \lambda}{\partial u^1}; \\
\Gamma_{12}^1 &= -\Gamma_{11}^2 = \Gamma_{22}^2 = \frac{1}{2} \frac{\partial \lambda}{\partial u^2}.
\end{align*}
\]
Thus the Mainardi-Codazzi equations reduce to
\[
\begin{align*}
\frac{\partial b_{11}}{\partial u^2} - \frac{\partial b_{12}}{\partial u^1} &= \frac{1}{2} (b_{11} + b_{22}) \frac{\partial \lambda}{\partial u^2} \quad \text{and} \\
\frac{\partial b_{22}}{\partial u^1} - \frac{\partial b_{12}}{\partial u^2} &= \frac{1}{2} (b_{11} + b_{12}) \frac{\partial \lambda}{\partial u^1}.
\end{align*}
\]

Now suppose that our surface \( S \) has constant mean curvature \( H \). Let \( z = u^1 + (-1)^{1/2} u^2 \), complex local coordinate, and define
\[
\phi(z) = (b_{11} - b_{22}) + 2 (-1)^{1/2} b_{12}.
\]
As \( 2H = b_{11} e^\lambda + b_{22} e^{-\lambda} = (b_{11} + b_{22}) e^\lambda \) is constant, (4.3) says that \( \partial/\partial \bar{z} = \frac{1}{2} \{\partial/\partial u^1 + (-1)^{1/2} \partial/\partial u^2\} \) annihilates \( \phi \); thus \( \phi \) is a holomorphic function of \( z \). Let \( f \) be the function on \( U \) defined by
\[
b_{11} = He^\lambda + f, \quad b_{22} = He^\lambda - f.
\]
Suppose that Gauss curvature satisfies
\[ K = H^2 - e^{-2\lambda}, \text{ i.e., } \lambda = -\frac{1}{2} \log(H^2 - K). \]

Then
\[ H e^{2\lambda} - 1 = K e^{2\lambda} = b_{11} b_{22} - b_{12}^2 = H e^{2\lambda} - (f^2 + b_{12}^2), \]
so \( f^2 + b_{12}^2 = 1 \). But \( \phi = 2(f + (-1)^{1/2} b_{12}) \) is holomorphic; now the maximum modulus principle says that \( \phi \) is constant; thus \( f \) and \( b_{12} \) are constant.

Notice that \( PC^2 \) by the assumption \( H^2 - K = e^{-2\lambda} > 0 \). Cutting \( U \) down if necessary, Theorem 3.1 gives us a local coordinate \( v \) on \( U \) in which \( I = e^\lambda \{ dv^1 \otimes dv^1 + dv^2 \otimes dv^2 \} \) and \( II = (He^\lambda + 1) dv^1 \otimes dv^1 + (He^\lambda - 1) dv^2 \otimes dv^2 \). If \( \alpha \) is the oriented angle from \( \partial/\partial u^1 \) to \( \partial/\partial v^1 \), the two expressions for \( I \) give
\[ dv^1 = \cos \alpha \, du^1 + \sin \alpha \, du^2 \text{ and } dv^2 = -\sin \alpha \, du^1 + \cos \alpha \, du^2. \]
Equating coefficients of \( du^1 \otimes du^1 \) in the two expressions for \( II \),
\[ He^\lambda + f = b_{11} = (He^\lambda + 1) \cos^2 \alpha + (He^\lambda - 1) \sin^2 \alpha \]
\[ = He^\lambda + \{ \cos^2 \alpha - \sin^2 \alpha \}. \]
Thus \( f = \cos^2 \alpha - \sin^2 \alpha = \cos(2\alpha) \). Similarly \( b_{12} = 2 \cos \alpha \sin \alpha = \sin(2\alpha) \).
Now \( \alpha \) is constant, and \( v^1 = \cos \alpha u^1 + \sin \alpha u^2 + c_1 \) and \( v^2 = -\sin \alpha u^1 + \cos \alpha u^2 + c_2 \) for some constants \( c_i \). We summarize as follows.

4.4. Theorem. Let \( S \) be a surface immersed in \( \mathbb{R}^3 \) with constant mean curvature \( H \), and define \( \lambda = -\frac{1}{2} \log(H^2 - K) \) on \( S_0 \). Let \( (U, u) \) be a connected local coordinate neighborhood such that \( U \subset S_0 \) and
\[ I = e^\lambda \{ du^1 \otimes du^1 + du^2 \otimes du^2 \}. \]
Then there is a constant \( \alpha \) such that
\[ II = (He^\lambda + \cos 2\alpha) du^1 \otimes du^1 + 2 \sin 2\alpha \, du^1 du^2 \]
\[ + (He^\lambda - \cos 2\alpha) du^2 \otimes du^2. \]
Let \( c_i \) be constants, \( v^1 = \cos \alpha u^1 + \sin \alpha u^2 + c_1 \) and \( v^2 = -\sin \alpha u^1 + \cos \alpha u^2 + c_2 \). Then \( v = (v^1, v^2) \) is a local coordinate on \( U \), \( \alpha \) is the angle from \( \partial/\partial u^1 \) to \( \partial/\partial v^1 \), and
\[ I = e^\lambda \{ dv^1 \otimes dv^1 + dv^2 \otimes dv^2 \} \text{ and } \]
\[ II = (He^\lambda + 1) dv^1 \otimes dv^1 + (He^\lambda - 1) dv^2 \otimes dv^2. \]

5. Obstruction to a congruence. The following result generalizes the fact that an isometry of small patches of a right circular cylinder is a congruence only when it preserves the direction of the axis of the cylinder.

5.1. Theorem. Let \( S, S' \) and \( S'' \) be connected surfaces embedded in
$\mathbb{R}^3$ with the same constant mean curvature $H$, which are not open subsets of a plane or a sphere. Then to every isometry $f: S \to S'$ we have a real number $\alpha(f)$, defined up to addition of an integral multiple of $\pi$, specified by the property: if $x \in S_a$ and $(U, u)$ is a local coordinate neighborhood of $x$ given by Theorem 3.1 then 

$$f^*II' = \{He^\lambda + \cos 2\alpha(f)\} du^1 \otimes du^1 + 2\sin 2\alpha(f) du^1 du^2 + \{He^\lambda - \cos 2\alpha(f)\} du^2 \otimes du^2$$

in $U$.

$\alpha$ has the properties:

(i) $f$ extends to a rigid motion of $\mathbb{R}^3$, if and only if $\alpha(f) \equiv 0 \mod \pi$;

(ii) $\alpha(f^{-1}) = -\alpha(f)$;

(iii) if $g: S' \to S''$ is an isometry, then $\alpha(g \circ f) = \alpha(f) + \alpha(g)$;

(iv) two isometries $f, g: S \to S'$ differ by a rigid motion of $\mathbb{R}^3$, if and only if $\alpha(f) \equiv \alpha(g) \mod \pi$.

Given $x \in S_a$ and a real number $b$, there is a neighborhood $V$ of $x$, a surface $W$ of constant mean curvature $H$, and an isometry $h: V \to W$, such that $\alpha(h) = b$.

The proof is based on the standard fact [2]:

5.2. Lemma. Let $S$ be a connected surface immersed in $\mathbb{R}^3$ with constant mean curvature. If some point of $S$ is not umbilic, then the umbilics of $S$ are isolated.

The function $\phi$ of §4 is holomorphic, and the umbilics of $S$ in the domain of $\phi$ are just the zeroes of $\phi$.

Now the surfaces $S, S'$ and $S''$ have all umbilics isolated; for an all-umbilic surface is an open subset of a plane or a sphere. Thus $S_a$ (resp. $S'_a$, resp. $S''_a$) is arcwise connected and dense in $S$ (resp. $S'$, resp. $S''$). Let $K, K'$ and $K''$ denote their Gauss curvature functions and let $f: S \to S'$ be an isometry. Then $f(S_a) = S'_a$ because $K'(f(x)) = K(x)$, $S_a = \{x \in S: H^2 - K(x) > 0\}$ and $S'_a = \{f(x) \in S': H^2 - K'(f(x)) > 0\}$. Similarly, the functions $\lambda = -\frac{1}{2} \log (H^2 - K)$ on $S_a$ and $\lambda' = -\frac{1}{2} \log (H^2 - K')$ on $S'_a$ are related by $\lambda = \lambda' \cdot f$.

Let $x \in S_a$ and choose a connected local coordinate neighborhood $(U, u)$ according to Theorem 3.1. Then $I = e^v \{du^1 \otimes du^1 + du^2 \otimes du^2\}$ and $II = (He^\lambda + 1) du^1 \otimes du^1 + (He^\lambda - 1) du^2 \otimes du^2$. Let $W = f(U), w^i(f(z)) = u^i(z)$ for $z \in U$; then $(W, w)$ is a connected local coordinate neighborhood of $f(x)$. $U \subset S_a$ implies $W \subset S''_a \cdot f^* I' = I$ because $f$ is an isometry, so $f^*I' = e^v \{dw^1 \otimes dw^1 + dw^2 \otimes dw^2\} = e^{\lambda'} \{d(w^1 \cdot f) \otimes d(w^1 \cdot f) + d(w^2 \cdot f) \otimes d(w^2 \cdot f)\}$; thus $I' = e^{\lambda'} \{dw^1 \otimes dw^1 + dw^2 \otimes dw^2\}$. Applying Theorem 4.4 to $S'$, we have a number $\alpha_f(x)$ such that $II' = He^\lambda + \cos 2\alpha_f(x) dw^1 \otimes dw^1 + 2\sin 2\alpha_f(x) dw^1 dw^2 + (He^\lambda - \cos 2\alpha_f(x)) dw^2 \otimes dw^2$. Thus $f^*II' = (He^\lambda + \cos 2\alpha_f(x)) du^1 \otimes du^1 + 2\sin 2\alpha_f(x) du^1 du^2 + (He^\lambda - \cos 2\alpha_f(x)) du^2 \otimes du^2$. This specifies $\alpha_f(x)$ up to an integral
multiple of $\pi$. The uniqueness part of Theorem 3.1 says that $\alpha_t(x)$ is well defined up to an integral multiple of $\pi$.

Let $C$ be the circle which is the real numbers modulo $\pi$. We have a map $\alpha_t: S_n \to C$. If $x \in S_n$ then $\alpha_t$ is constant on a neighborhood of $x$. As $S_n$ is connected, now $\alpha_t$ is constant. Let $\alpha(f)$ denote its value. We have proved the existence of a number $\alpha(f)$ defined modulo $\pi$ and specified by $f^{*}I''$ as required.

If $\alpha(f) \equiv 0 \mod \pi$ if and only if $\cos 2\alpha(f) = 1$ and $\sin 2\alpha(f) = 0$, which is equivalent to $f^{*}I'' = I$. In that case $f: S \to S'$ is a diffeomorphism of connected surfaces in $R^3$ such that $f^{*}I' = I$ and $f^{*}I'' = I$, and a classical theorem says that $f$ extends to a rigid motion of $R^3$. This proves (i).

For (iii) we have a local coordinate $v$ on $W \subset S'$ given by $(\alpha_t = \alpha(f))$

$v^1 = \cos(\alpha_t)w^1 + \sin(\alpha_t)w^2$ and $v^2 = -\sin(\alpha_t)w^1 + \cos(\alpha_t)w^2$, and $I'' = (He^{*})^{\lambda} + 1)dv^1 \otimes dv^1 + (He^{*} - 1)dv^2 \otimes dv^2$ on $W$. Let $X = g(W) \subset S''_n$ and let $x$ be the local coordinate on $X$ with $x \cdot g = v$. Define $y$ on $X$ by $y^1 = \cos(\alpha_g)x^1 + \sin(\alpha_g)x^2$ and $y^2 = -\sin(\alpha_g)x^1 + \cos(\alpha_g)x^2$; then $I''' = (He^{*} - 1)dy^1 \otimes dy^1 + (He^{*} - 1)dy^2 \otimes dy^2$ on $X$. We compute

$u^1 = \cos(\alpha_t + \alpha_g)(y^1 \cdot g \cdot f) + \sin(\alpha_t + \alpha_g)(y^2 \cdot g \cdot f)$.

$u^2 = -\sin(\alpha_t + \alpha_g)(y^1 \cdot g \cdot f) + \cos(\alpha_t + \alpha_g)(y^2 \cdot g \cdot f)$.

Thus $\alpha_{g \cdot f} = \alpha_g + \alpha_f$.

Now (ii) and (iv) are immediate.

Let $x \in S_n$ and $b \in R$. Choose a local coordinate neighborhood $(U, u)$ of $x$ as in Theorem 3.1. $D = u(U) \subset R^2$ is the parameter domain. We define functions $g_{11}(u(z)) = g_{22}(u(z)) = e^{\lambda(z)}$, $g_{12} = g_{21} = 0$, $b_{11}(u(z)) = He^{\lambda(z)} + \cos(2b)$, $b_{22}(u(z)) = He^{\lambda(z)} - \cos(2b)$ and $b_{12}(u(z)) = b_{21}(u(z)) = \sin(2b)$. Then the forms

$I_0 = \sum g_{ij}(u)du^i \otimes du^j$ and $II_0 = \sum b_{ij}(u)du^i \otimes du^j$

satisfy the Mainardi-Codazzi equations (4.3). As $S$ satisfies the Gauss equation a priori, and as $I_0 = I$ and $\det (b_{ij}) = H^2e^{2\lambda} - 1$ as for $S$, now $I_0$ and $II_0$ satisfy the Gauss equation. Thus Bonnet’s existence theorem says that every $m \in D$ has a neighborhood $V(m) \subset D$ which is parameter domain for a local surface $W \subset R^3$ with first and second fundamental forms $I_0$ and $II_0$. Let $w$ be the coordinate on $W$, $m = u(x)$, $V = u^{-1}(V(m))$. Then $h = u^{-1} \cdot u \mid V$ is a diffeomorphism of $V$ onto $W$ such that $h^{*}I_0 = I_0 = I$ (so $h$ is an isometry) and $h^{*}II_0 = II_0$ (so $\alpha(h) = b$). q.e.d.

Theorem 5.1 requires $S_n$ to be nonempty. Thus we remark:

5.3. Complement to Theorem 5.1. Let $S$ and $S'$ be connected all-
umbilic surfaces in $R^3$. Then any isometry $f: S \to S'$ extends to a rigid motion of $R^3$.

For $f$ is a diffeomorphism with $f^*I' = I$, and we need only check that $f^*I' = I$. An all-umbilic surface is an open subset of a plane or a sphere. In the first case $I = 0$ and $K = 0$. In the second case $I = \frac{1}{r}I$ and $K = 1/r^2$, where $r$ is the radius of the sphere. As $f$ preserves Gauss curvature, now $f^*I' = I$. q.e.d.

6. The Gauss Equation. The Gauss equation for a surface $S$ with first fundamental form $\epsilon \left\{ du^1 \otimes du^1 + du^2 \otimes du^2 \right\}$ says that Gauss curvature is given by:

$$K = \frac{-1}{2} e^{-\lambda}\Delta \lambda, \quad \Delta = \frac{\partial^2}{\partial u^1 \partial u^1} + \frac{\partial^2}{\partial u^2 \partial u^2}.$$ 

Now suppose that $S$ has constant mean curvature $H$ and that our local coordinate neighborhood $(U, u)$ is given by Theorem 3.1. Then $K = H^2 - e^{-2\lambda}$, so (6.1) becomes a nonlinear elliptic equation

$$\Delta \lambda = 2(e^{-\lambda} - H^2 e^\lambda).$$

We regard this as an equation for $K = H^2 - e^{-2\lambda}$.

We regard (6.2) as a boundary value problem on a disc $D$ of radius $r > 0$ in $R^2$. Let $b$ be a continuous function on the boundary $\partial D$ of the disc; we look for a solution $\lambda(u^1, u^2)$ to (6.2) on $D$, continuous on the closure and with values $b$ on $\partial D$. Let $h(u^1, u^2)$ be the harmonic function on $D$ with boundary values $b$. Then we write (6.2) in the form (this defines $F$)

$$\Delta \eta = 2(e^{-\lambda} - H^2 e^{\lambda + h}) \equiv F(\eta), \quad \eta = \lambda - h,$$

and we want a solution $\eta$ on $D$ vanishing on $\partial D$.

Following Courant-Hilbert ([1, Appendix to Chapter 4]), such solutions $\eta$ exist provided that certain bounds $c, m$ satisfy $(r + r^2)cm \leq 1/4$. Here $c$ and $m$ are defined as follows. Let $C^2(D)$ denote the set of all continuous functions on the closure of $D$ which are twice differentiable in $D$. We remark that this has an interesting expression in complex notation. There $z = u^1 + (-1)^{1/2}u^2$ is the variable, so $ds = du^1 + (-1)^{1/2}du^2$ and $dz = du^1 - (-1)^{1/2}du^2$, and the vector fields dual to these forms are $\partial / \partial z = (1/2) \{ \partial / \partial u^1 + (-1)^{1/2} \partial / \partial u^2 \}$ and $\partial / \partial s = (1/2) \{ \partial / \partial u^1 + (-1)^{1/2} \partial / \partial u^2 \}$. The exterior derivative $d = d^1 + d^2$ where by definition $d^1 = \partial f / \partial s ds$ and $d^2 = \partial f / \partial z dz$ on functions. In particular $d^1 d^2 f = \frac{1}{2} \Delta f dz \wedge ds$. The element of area on $S$ is given by $dA = e^{\lambda} du^1 \wedge du^2 = (1/2) (-1)^{1/2} e^\lambda \cdot ds \wedge dz$. Thus (6.1) can be written in coordinate free form $KdA = - (1)^{1/2} d^{1/2} \lambda$.

Writing it out in terms of $K$, one obtains $\Delta K = \{ (\partial K / \partial u^1)^2 + (\partial K / \partial u^2)^2 \} (H^2 - K) + 4 (H^2 - K)^{1/2} - 4H^2 (H^2 - K)^{1/2}, K < H^2$, which is more difficult to study than is (6.2).
continuously differentiable in $D$. Then $c$ is specified by
\[
\max \left| \frac{\partial f}{\partial u^i} \right| \leq cr \text{ l.u.b. } |\Delta f| \quad \text{for } f \in C^2(D),
\]
and $c$ is independent of choice of $r$ or $D$. Define the norm
\[
||f|| = \max |f| + \sum \max \left| \frac{\partial f}{\partial u^i} \right|
\]
on $C^2(D)$. Then $m$ is any common bound for all $|F(f)|$ and all $|dF/df|$ with $||f|| \leq 1$. A glance at the form of $F$ in (6.3) shows now that $m$ is any common bound for $2(e^{-1-h} + H^2 e^{1+h})$ and $2(e^{1-h} + H^2 e^{h-1})$. As $h$ achieves its maximum on $\partial D$, now we may take
\[
(6.4) \quad m = \max \{2(e^{-1-\beta} + H^2 e^{1+\beta}), 2(e^{1-\beta} + H^2 e^{\beta-1})\}, \quad \beta = \max |b|.
\]
In summary, and using the fact that solutions to elliptic equations are analytic,

6.5. Theorem. Let $D$ be a disc of radius $r > 0$ in $\mathbb{R}^2$, let $b$ be a continuous function on $\partial D$, and let $B = \max_{\partial D} |b|$. If
\[
(6.6) \quad (r + r^2)^{-1} \geq 8c \cdot \max \{e^{-1-\beta} + H^2 e^{1+\beta}, e^{1-\beta} + H^2 e^{\beta-1}\},
\]
then there exists a continuous function $\lambda$ on the closure of $D$, real analytic on $D$, which satisfies (6.2) and has values $b$ on $\partial D$.

The usual uniqueness condition for an equation $\Delta f = A(f, u)$ for given boundary values is $\partial A/\partial f \geq 0$. But in our case (6.2) we always have $\partial A/\partial f < 0$.

We can now describe the construction of umbilic-free local surfaces of constant mean curvature. Such a surface is specified up to congruence by its first and second fundamental forms, and the condition for two candidates
\[
I = \sum g_{ij} du^i \otimes du^j \quad \text{and} \quad II = \sum b_{ij} du^i \otimes du^j
\]
to give a surface, is that the $g_{ij}$ and $b_{ij}$ satisfy the Mainardi-Codazzi equations and the Gauss equation.

If $H$ is the mean curvature of the desired surfaces, then Theorem 3.1 allows us to take $I$ and $II$ in the forms
\[
I = e^{\lambda} \{ du^1 \otimes du^1 + du^2 \otimes du^2 \} \quad \text{and}
\]
\[
II = \{ He^{\lambda} + \cos(2t) \} du^1 \otimes du^1 + 2 \sin(2t) du^1 du^2
\]
\[
+ \{ He^{\lambda} - \cos(2t) \} du^2 \otimes du^2
\]
for $t = 0$. For any constant $t$, the Mainardi-Codazzi equations (4.3) are satisfied for $I$ and $II_t$, and the Gauss equation (6.2) is

$$\Delta \lambda = 2(e^{-\lambda} - H^2 e^\lambda), \quad K = H^2 - e^{-2\lambda}.$$  

Theorem 6.5 gives the existence of many solutions. Given a local solution $\lambda$, there corresponds a well defined congruence class of local surfaces $S_{\lambda, t}$ with $I$ as specified and $II = II_t$; Theorem 5.1 shows that the natural map $^7 f_{*, t}: S_{\lambda, s} \rightarrow S_{\lambda, t}$ is an isometry and $\alpha(f_{*, t}) = t - s$.

**References**


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$^7 f_{*, t}$ is given by $u^t(f_{*, t}(x)) = u^t(x)$. 

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