NOTE ON A WHITEHEAD PRODUCT

GADE V. KRISHNARAO

The aim of this note is to prove

THEOREM. *Let ι_n generate $\pi_n(S^n)$ and $n=4s+1$, s being a positive integer. Then $[\iota_n, \iota_n]$ is not divisible by two.*

Let i, j, and k be homomorphisms in the exact sequences of the fiber bundles $O(n)\to O(n+1)\to S^n$, $n\geqq 1$

$$\text{(A)} \qquad \cdots \pi_m(O(n)) \xrightarrow{i} \pi_m(O(n+1)) \xrightarrow{j} \pi_m(S^n) \xrightarrow{k} \pi_{m-1}(O(n))$$

and $d=j\circ k$ be the composite

$$\pi_{m+1}(S^{n+1}) \xrightarrow{k} \pi_m(O(n+1)) \xrightarrow{j} \pi_m(S^n).$$

We state an easy lemma without proof.

LEMMA 1. *Let $(E, F)\to(B, *)$ be a fibration and ∂ be the boundary homomorphism in its exact homotopy sequence. Then $\partial(\alpha\circ E\beta)=\partial(\alpha)\circ\beta$ where $\alpha\in\pi_r(B)$, $E\beta\in\pi_q(S^r)$ and E is the suspension homomorphism.*

LEMMA 2. *For odd n, $[\iota_n, \iota_n]$ is divisible by two if and only if it is in the image of d.*

PROOF. It is known [3, p. 120] that $d(\iota_{n+1})=2\iota_n$ if n is odd. Since $E\colon \pi_{2n-1}(S^n)\to\pi_{2n}(S^{n+1})$ is onto and its kernel is generated by $[\iota_n, \iota_n]$ and also $E\colon \pi_{2n-2}(S^{n-1})\to\pi_{2n-1}(S^n)$ is onto, we have, for any element $E\alpha$ in $\pi_{2n}(S^{n+1})$

$$\begin{aligned} d(E\alpha) &= j\circ k(E\alpha) = j\circ k(\iota_{n+1}\circ E\alpha) \\ &= 2\iota_n\circ\alpha \qquad \text{by Lemma 1} \\ &= 2\alpha. \end{aligned}$$

If $[\iota_n, \iota_n]=2\alpha$, for some α in $\pi_{2n-1}(S^n)$, then $E\alpha\neq 0$ and $d(E\alpha)=2\alpha$. That proves the lemma.

If $n=4s+1$, S^n admits only a 1-field. Consequently [2], (i) there exists $b\in\pi_{n-1}(O(n-1))$ such that $i(b)=k(\iota_n)$ and $j(b)=\eta$ where η is the stable element in $\pi_{n-1}(S^{n-2})$ (we denote all suspensions of η by the same symbol), and (ii) $[\iota_n, \iota_n]=E\theta$ where $\theta\in\pi_{2n-2}(S^{n-1})$ is not a suspension.

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$$j \circ (i)^{-1} \circ k(E\theta) = \eta \circ \theta \qquad \text{by Lemma 1.}$$

Consider the generalized Hopf invariant

$$H: \pi_{2n-2}(S^{n-2}) \longrightarrow \pi_{2n-2}(S^{2n-5}),$$
$$\begin{aligned} H(\eta \circ \theta) &= E(\eta \# \eta) \circ H(\theta), && \text{see [4, p. 18]} \\ &= \eta \circ \eta \circ H(\theta), && \text{see [1]} \\ &= \eta \circ \eta \circ \eta \neq 0 \end{aligned}$$

because θ is not a suspension, $H(\theta) \neq 0$ and hence $H(\theta) = \eta$. Therefore $\eta \circ \theta \neq 0$ and $k(E\theta) \neq 0$ and the exactness of (A) implies that $E\theta = [\iota_n, \iota_n]$ is not in the image of $d = j \circ k$. This proves the Theorem.

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INDIAN INSTITUTE OF TECHNOLOGY, KANPUR