SEMICHARACTERS ON SUBSEMIGROUPS OF AN ABELIAN TOPOLOGICAL GROUP

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In this note we prove a generalization of a theorem of A. Gleason [**3**, p. 60] on continuous semicharacters of semigroups contained in abelian topological groups.

THEOREM 1 (GLEASON). *Let G be an abelian topological group, S a subsemigroup of G, and t an element of the interior of S, but $-t\notin S$. For each complex number z of modulus $\leqq 1$, there is a continuous multiplicative function χ on S to the complex unit disk* (semicharacter *of* S) *with* $\chi(t)=z$.

The form of this theorem credited to Gleason in [**3**] applies only to locally compact groups G and depends somewhat on the structure of locally compact abelian groups. As observed in [**3**], this theorem guarantees that the harmonic analysis of semigroups with nonempty interior in G, which are not subgroups, does not collapse to the ordinary theory of characters of G.

The proof given here is based on a version of the Hahn-Banach theorem for semigroups, [**1**].

EXTENSION THEOREM. *Let S be an abelian semigroup, and ω a real function on S subject to the conditions*

(1) $-\infty\leqq\omega(s)<\infty,\ s\in S.$

(2) $\omega(s_1+s_2)\leqq\omega(s_1)+\omega(s_2),\ s_i\in S.$

Let H be a subsemigroup of S and ϕ a real function on H such that

(3) $-\infty\leqq\phi(h)<\infty,\ h\in H.$

(4) $\phi(h_1+h_2)=\phi(h_1)+\phi(h_2),\ h_i\in H.$

(5) $\phi(s+h)\leqq\omega(s)+\phi(h)$, *whenever* $s\in S,\ h\in H,\ s+h\in H.$

Then, ϕ can be extended to a function ξ on S for which

(6) $-\infty\leqq\xi\leqq\omega.$

(7) $\xi(s_1+s_2)=\xi(s_1)+\xi(s_2),\ s_i\in S.$

A special case, and the only one presently necessary, is that for each t in S there is some ξ satisfying (5) and (6) with $\xi(t)=\lim_{n\to\infty}\omega(nt)/n\equiv\omega_\infty(t)$. For we have only to set $\phi(nt)=n\omega_\infty(t)$ for $n\geqq 1$ so that if $s+kt=mt$, $s\in S$, m, $k\geqq 1$,

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$$\begin{aligned}\phi(s + kt) &= m \lim_{n\to\infty} \frac{1}{n}\,\omega(nt) = \lim_{n\to\infty} \frac{1}{n}\,\omega(mnt)\\ &= \lim_{n\to\infty} \frac{1}{n}\,\omega(ns + nkt) \leqq \lim_{n\to\infty}\left[\frac{1}{n}\,\omega(ns) + \frac{1}{n}\,\omega(nkt)\right]\\ &\leqq \omega(s) + k\omega_\infty(t) = \omega(s) + \phi(kt).\end{aligned}$$

This special case can be derived also from Beurling's formula for the spectral radius in a commutative Banach algebra, [2].

Proof of Theorem 1. For $s \in S$ define

$$\omega(s) = \inf\{m\colon s + mt \in S,\ m \text{ an integer}\}.$$

Clearly $-\infty \leqq \omega(s) \leqq 0$ and ω is subadditive as in (1), (2) since $S+S\subseteq S$. Inasmuch as $nt = t+(n-1)t$ and $nt \notin S+(n-1)t$, $n \geqq 1$, $-n \leqq \omega(nt) \leqq 1-n$; $\omega(nt) = -n$ precisely when $0 \in S$. In any case, $\omega_\infty(t) = -1$. Applying the Extension Theorem we obtain an additive function ξ on S such that $-\infty \leqq \xi \leqq \omega$ and $\xi(t) = -1$.

ξ is continuous at t, for if V is a symmetric neighborhood of 0 in G such that $t+nV\subseteq S$, then $nt+nV \subseteq (n-1)t+S$. Now $\xi \leqq \omega \leqq 1-n$ on the subset $nt+nV$ of S; $\xi \leqq 1/n-1$ on $t+V$. Let $v \in V$ and observe that $-2 = \xi(2t) = \xi(t+v)+\xi(t-v) \leqq \xi(t+v)+1/n-1$: $\xi(t+v) \geqq -1 -1/n$. In general, when $s\in S$, $v \in V$, $s+v\in S$, $t+v \in S$, we can write $\xi(s+v)+\xi(t) = \xi(s+v+t) = \xi(s)+\xi(t+v)$. This proves that ξ is continuous on S and that Z, the subset of S on which $\xi = -\infty$, is open and closed; moreover $S+Z\subseteq Z$.

For λ a complex number with $\operatorname{Re}\lambda \geqq 0$ define $\chi(s) = \exp[\lambda\xi(s)]$ if $s \notin Z$ and $\chi(Z) = 0$. Then $|\chi| \leqq 1$ in S and $\chi(t) = e^{-\lambda}$. χ is multiplicative and continuous because Z is an open-closed ideal of S. This completes the proof.

When G is locally compact and S measurable, the map $\lambda \to \exp[\lambda\xi]$ defines an analytic disk in the maximal ideals of certain Banach algebras.

References

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3. M. A. Rieffel, *A characterization of commutative group algebras and measure algebras*, Trans. Amer. Math. Soc. **116** (1965), 32–65.

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