THE LOCAL FINITE-AREA PRINCIPLE IN THE HALF-PLANE[1]

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1. The familiar finite-area principle of Fejér asserts that if the image of $|z|<1$ under the analytic mapping $w=f(z)=\sum_0^\infty a_n z^n$ is of finite area (counting multiplicities), then $\sum a_n z^n$ converges a.e. on $|z|=1$ and uniformly on closed arcs of continuity. This result was localized by Zygmund [3] and by Lusin [1] who considered the image of a region bounded by a simple Jordan arc in $|z|<1$ and an arc $\alpha \leqq \theta \leqq \beta$ of $|z|=1$. They showed that if $a_n=o(1)$ then the conclusions of the Fejér theorem hold relative to the arc $[\alpha, \beta]$ and, for $a_n=o(n^k)$, $k>-1$, convergence can be replaced by (C, k) summability. It should be noted that the Tauberian conditions in this result are necessary in order that there be a point of convergence (or (C, k) summability) on $|z|=1$.

The result we will establish is a localized finite area theorem for functions analytic in a half-plane.

THEOREM. *Let* $f(s)=\int_0^\infty e^{-sx}d\gamma(x)$, *where* $s=\sigma+i\tau$, *be analytic in the half-plane* $\sigma>0$. *Suppose that*

$$(\mathrm{A}_0) \qquad \alpha(x) = \sup_{0\leqq h\leqq 1} |\gamma(x+h)-\gamma(x)| = o(1).$$

Let Ω *be a region in* $\sigma>0$ *bounded by a segment* $[i\alpha, i\beta]$ *of* $\sigma=0$ *and a Jordan arc. If*

$$\int_\Omega\int |f'|^2 d\sigma d\tau < \infty,$$

Then $\int_0^\infty e^{-sx}d\gamma(x)$ *converges a.e. on the segment* $(i\alpha, i\beta)$ *and uniformly on any closed subsegment of continuity. If* (A_0) *is replaced by*

$$(\mathrm{A}_k) \qquad \alpha(x) = o(x^k), \qquad k>0,$$

then convergence is replaced by (C, k) *summability in the conclusion.*

It should be noted that (A_0) is a necessary condition for convergence of the integral at one point of $\sigma=0$, but (A_k), contrary to [5, p. 335] is not necessary for (C, k) summability. A counterexample is given in §3.

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2. We turn now to the proof of the theorem. We consider only the case of (C, k) summability. Clearly there will be no loss of generality if we assume that $\gamma(x) \equiv 0$ for $0 \leqq x < \delta$ and that $0 < \alpha < \beta < 2\pi$.

Let γ_1 and γ_2 be the odd and even extensions of $\mathcal{R}(\gamma)$ and $\mathcal{I}(\gamma)$ to $(-\infty, \infty)$. Set $\phi = (\gamma_1 + i\gamma_2)/2$ and $t = -\tau$. Proceeding formally we have

$$\int_0^\infty e^{-i\tau x}\, d\gamma(x) = \int_{-\infty}^\infty e^{itx}\, d\phi(x) + i\int_{-\infty}^\infty e^{itx}(-i \operatorname{sign} x)\, d\phi(x).$$

Clearly ϕ satisfies condition (A_k) and, by an integration by parts, it may be seen that

$$\psi(x) = \int_0^x y\, d\phi(y)$$

satisfies condition (A_{k+1}).

Let h and p denote positive integers which will be chosen as large as is needed. We define

$$\Phi_h(x) = \int_{-\infty}^x (iy)^{-h}\, d\phi(y), \qquad \Psi_h(x) = \int_{-\infty}^x (iy)^{-h}\, d\psi(y)$$

and

$$F(t) = \int_{-\infty}^\infty e^{itx}\, d\Phi_{h-1}(x), \qquad F^*(t) = \int_{-\infty}^\infty e^{itx}\, d\Psi_h(x).$$

Clearly $\Psi_h = -i\Phi_{h-1}$ and so $F^*(t) = -iF(t)$.

Let $\lambda(t)$ be a function of period 2π and in class C^p such that

$$\begin{aligned} \lambda(t) &= 1 \quad \text{for } \alpha \leqq t \leqq \beta, \\ &= 0 \quad \text{for } 0 < t < a < \alpha \quad \text{and} \quad \beta < b < t < 2\pi. \end{aligned}$$

Let us now consider the formal hth derivative of the Fourier series of $F^*\lambda$, $\mathcal{S}^{(h)}(F^*\lambda) = \sum_{-\infty}^{\infty} \beta_n e^{int}$ with $\beta_n = o(n^{k+1})$. Clearly $\mathcal{S}(F\lambda) = -i\mathcal{S}(F^*\lambda)$ and so

$$\mathcal{S}^{(h-1)}(F\lambda) = \sum_{-\infty}^{\infty} b_n e^{int}$$

with $b_n = o(n^k)$ and $\beta_n = -nb_n$. Let

$$g(z) = \sum_{-\infty}^{\infty} b_n r^{|n|} e^{int} + i \sum_{-\infty}^{\infty} (-i \operatorname{sign} n) b_n r^{|n|} e^{int} = \sum_0^\infty c_n z^n$$

where $c_n = 2b_n$, $z = re^{it}$. Then

$$\sum_{-\infty}^{\infty} nb_n r^{|n|} e^{int} + i \sum_{-\infty}^{\infty} (-i \text{ sign } n) nb_n r^{|n|} e^{int} = \sum_{0}^{\infty} 2b_n n z^n = zg'(z).$$

Applying the method employed by Zygmund in [**4**, Theorem 9] to the function ψ we see that, as $\omega \to \infty$, the differences

$$\int_{-\omega}^{\omega} xe^{itx}\, d\phi(x) - \sum_{|n| \leq \omega} \beta_n e^{int}$$

$$\int_{-\omega}^{\omega} xe^{itx}(-i \text{ sign } x)\, d\phi(x) - \sum_{|n| \leq \omega} \beta_n (-i \text{ sign } n) e^{int}$$

are uniformly $(C, k+1)$ summable in $[\alpha, \beta]$, the first difference to zero and the second to a finite value.

We observe now that

$$\mathcal{R}(f'(s)) = -\int_{-\infty}^{\infty} e^{-\sigma|x|} e^{itx} |x|\, d\phi(x);$$

$$\mathcal{I}(f'(s)) = \int_{-\infty}^{\infty} e^{-\sigma|x|} e^{itx} ix\, d\phi(x).$$

Since the uniform Cesàro summability implies uniform Abel summability we have

$$i\mathcal{I}(f'(s)) - \sum_{-\infty}^{\infty} nb_n e^{int} r^n \to 0,$$

$$i\mathcal{R}(f'(s)) - \sum_{-\infty}^{\infty} nb_n (i \text{ sign } n) e^{int} r^{|n|} \to \text{finite value}$$

uniformly on $[\alpha, \beta]$ as $\sigma = -\log r \to 0+$. Hence for some $\epsilon > 0$ there is an $M > 0$ such that

$$|f'(s)|^2 + M \geqq |zg'(z)|^2 > 1/2\, |g'(z)|^2$$

for $\alpha \leqq t \leqq \beta$ and $0 < \sigma = -\log r < \epsilon$.

Thus there exists an $M' > 0$ and a region Ω' in $|z| < 1$ bounded by the arc $\alpha \leqq t \leqq \beta$ of $|z| = 1$ and a simple Jordan arc in $|z| < 1$ such that

$$\int_{\Omega'}\int |g'(z)|^2 r\, dr\, dt \leqq M' + 2 \int_{\Omega}\int |f'(s)|^2\, d\sigma\, d\tau < \infty.$$

Since $c_n = o(n^k)$, the localized finite area theorem of Zygmund is applicable. Thus $\sum_0^\infty b_n e^{int}$ is (C, k) summable a.e. on (α, β) and uniformly on closed subarcs of continuity. This implies the same for $\sum_{-\infty}^{\infty} b_n e^{int}$ and $\sum_{-\infty}^{\infty} (-i \text{ sign } n) b_n e^{int}$.

If we now apply the method of Zygmund to the function ϕ satisfying condition (A_k) we find that the differences

$$\int_{-\omega}^{\omega} e^{itx}\,d\phi(x) - \sum_{|n|\leq\omega} b_n e^{int},$$

$$\int_{-\omega}^{\omega} e^{itx}(i \operatorname{sign} x)\,d\phi(x) - \sum_{|n|\leq\omega} b_n(i \operatorname{sign} n)e^{int}$$

are uniformly summable (C, k) in $[\alpha, \beta]$, the first to zero and the second to a finite value. The summability properties of the integrals are then the same as those of the series, which establishes the theorem.

3. Consider now the function

$$\begin{aligned}\gamma(x) &= 0 \quad \text{if } n \leqq x \leqq n+1-1/2^n,\\ &= 2^n \quad \text{if } n+1-1/2^n < x < n+1\end{aligned}$$

for $n=0, 1, 2, \cdots$. Then

$$(C, 1)\int_0^{\infty} d\gamma(x) = \lim\,(1/\omega)\int_0^{\omega}\int_0^{u} d\gamma(x)\,du$$

$$= \lim\,(1/\omega)\int_0^{\omega}\gamma(u)\,du = 1$$

since

$$[\omega] \leqq \int_0^{\omega}\gamma(u)\,du \leqq [\omega]+1.$$

But

$$\sup_{0\leqq h\leqq 1}\left|\gamma(x+h)-\gamma(x)\right| = 2^{[x]} \neq o(x).$$

Hence $\int_0^\infty e^{-sx}d\gamma(x)$ is $(C, 1)$ summable at $s=0$, but $\gamma(x)$ does not satisfy condition (A_1) contrary to [**5**, p. 335].

References

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