MULTIPLICATION THEOREMS ON STRONGLY SUMMABLE SERIES

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1. Introduction.

1.1. Let $\{\lambda_n\}$ be an arbitrary increasing sequence of positive numbers, such that

$$0 < \lambda_0 < \lambda_1 < \lambda_2 < \cdots < \lambda_{n-1} < \lambda_n \to \infty, \quad \text{as} \quad n \to \infty,$$

and $\sum_{n=0}^{\infty} a_n$ a given series.

We write

$$A_n = a_0 + a_1 + a_2 + \cdots + a_{n-1} + a_n, \quad A_\lambda(\omega) = 0, \quad \text{for } \omega \leqq h,$$

where h is a convenient positive number.

If $\omega > 0$, $\lambda_n < \omega < \lambda_{n+1}$ then

$$A_\lambda(\omega) = A_n = \sum_{v=0}^{n} a_v = \sum_{\lambda_v < \omega} a_v$$

and for $k > 0$

$$\begin{aligned} A_\lambda^k(\omega) &= \sum_{\lambda_n < \omega} (\omega - \lambda_n)^k a_n \\ &= k \int_0^\omega (\omega - t)^{k-1} A_\lambda(t)\, dt = \int_0^\omega (\omega - t)^k\, dA_\lambda(t). \end{aligned}$$

We define $A_\lambda^0(\omega) = A_\lambda(\omega)$. We also define

$$\begin{aligned} \bar{A}_\lambda^k(\omega) &= \sum_{\lambda_n < \omega} (\omega - \lambda_n)^{k-1} \lambda_n a_n & (k > 0) \\ &= -\int_0^\omega A_\lambda(t) \frac{d}{dt} [(\omega - t)^{k-1} t]\, dt & (k > 1) \\ &= \int_0^\omega (\omega - t)^{k-1} t\, dA_\lambda(t) & (k \geqq 1). \end{aligned}$$

We have

Received by the editors October 12, 1964 and, in revised form, August 30, 1965 and February 8, 1966.

[1] This research has been supported by a Junior Research Fellowship of The Council of Scientific and Industrial Research, India.

$$\frac{d}{d\omega}\left(\frac{\overset{k}{A}_\lambda(\omega)}{\omega^k}\right) = \frac{k}{\omega^{k+1}}\overset{k}{\overline{A}}_\lambda(\omega). \tag{1.1}$$

We use $B_\mu^k(\omega)$, $\overline{B}_\mu^k(\omega)$ and $C_\nu^k(\omega)$, $\overline{C}_\nu^k(\omega)$ for similar expressions involving $\sum_{n=0}^\infty b_n$ and $\sum_{n=0}^\infty c_n$ respectively.

1.2. If we associate summability by Riesz means of type λ with the series $\sum_{n=0}^\infty a_n$ and type μ with $\sum_{n=0}^\infty b_n$, we may form the sequence of numbers ν_n, which are numbers $\lambda_p+\mu_q$ arranged in increasing order of magnitude, and associate summability by Riesz means of type ν with the series $\sum_{n=0}^\infty c_n$, where

$$c_n = \sum_{\lambda_p+\mu_q=\nu_n} a_p b_q.$$

Then we call $\sum_{n=0}^\infty c_n$ the Dirichlet product of $\sum_{n=0}^\infty a_n$ and $\sum_{n=0}^\infty b_n$. If $\lambda_n=\mu_n=n$, then the rule reduces to Cauchy's.

2. **Definitions.** The series $\sum_{n=0}^\infty a_n$ is said to be summable (R, λ, k), where $k \geqq 0$, to the sum s if

$$\lim_{\omega\to\infty} \overset{k}{A}_\lambda(\omega)/\overset{k}{\omega} = s \qquad \text{(cf. [3])}.$$

If, in addition,

$$\int_h^\omega \left| u \frac{d}{du}\left(\frac{\overset{k}{A}_\lambda(u)}{u^k}\right)\right|^r du = o(\omega), \tag{2.1}$$

as $\omega\to\infty$, then the series $\sum_{n=0}^\infty a_n$ is said to be summable $[R, \lambda, k, r]$ to the sum s, $(k>0, r\geqq 1, k>1/r')$, where r' denotes the number conjugate to r, i.e. $r'=r/(r-1)$ [5]. We define r' to be ∞ if $r=1$.

For the definition to be valid at all, the condition $k>1/r'$ is essential as pointed out by Boyd and Hyslop [2, pp. 94–95].

When $r=1$, $[R, \lambda, k]$ and $[R, \lambda, k, r]$ denote the same method. Now $[R, \lambda, 0]$ summability is equivalent to convergence and

$$\int_h^X x\,|\,d\overset{0}{A}_\lambda(x)\,| = o(X), \quad \text{as} \quad X\to\infty.$$

The above condition is the same as

$$\sum_{\lambda_n<X} |\,a_n\lambda_n\,| = o(X)$$

[5]. We observe that on account of (1.1) the condition (2.1) is equivalent to

$$(2.2)\qquad \int_h^X \left| \frac{\overline{A}_\lambda^k(u)}{u^k} \right|^r du = o(X), \quad \text{as} \quad X \to \infty.$$

Again, since $h>0$ and $\overline{A}_\lambda^k(u)$ is integrable (L) in the range (h, X) for every finite $X>h$, the condition (2.2) is equivalent to

$$(2.3)\qquad \int_h^X \left| \overline{A}_\lambda^k(u) \right|^r du = o(X^{kr+1}), \quad \text{as} \quad X \to \infty \quad [\mathbf{5}].$$

The assertion that the series $\sum_{n=0}^{\infty} a_n$ is summable $|R, \lambda, 0|$ to s means that $\sum_{n=0}^{\infty} a_n = s$ (in the usual sense) and $\sum_{n=0}^{\infty} |a_n| < \infty$.

It has been shown by Srivastava [**5**, p. 68, Theorem 9 and p. 61, Theorem 1] that, for $k \geqq 0$, summability $|R, \lambda, k|$ implies summability $[R, \lambda, k]$ and so also summability (R, λ, k).

3. The following theorems are known.

THEOREM 1. *If $\sum_{n=0}^{\infty} a_n$ is summable (R, λ, k) to sum s, $k \geqq 0$, and $\sum_{n=0}^{\infty} b_n$ is summable (R, μ, l) to sum t, then $\sum_{n=0}^{\infty} c_n$ is summable $(R, \nu, k+l+1)$ to sum st, $(l \geqq 0)$.*

THEOREM 2. *If $\sum_{n=0}^{\infty} a_n$ is summable $[R, \lambda, k]$, $k>0$, to sum s and $\sum_{n=0}^{\infty} b_n$ is summable (R, μ, l) to sum t, then the series $\sum_{n=0}^{\infty} c_n$ is summable $(R, \nu, k+l)$ to sum st.*

THEOREM 3. *If $\sum_{n=0}^{\infty} a_n$ is summable $[C, k]$, where $k>0$, to s and $\sum_{n=0}^{\infty} b_n$ is summable $|C, 0|$ to t, then $\sum_{n=0}^{\infty} (a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0)$ is summable $[C, k]$ to st.*

Theorems 1 and 2 are due to Chandrasekharan and Minakshisundaram [**3**, p. 100, Corollary 3.91 and p. 106, Theorem 3.96]. Theorem 3 has recently been obtained by A. V. Boyd [**1**]. We obtain in Theorem A the analogue of Theorem 3 for the Dirichlet product. Theorem B is concerned with summability $[R, \lambda, k, r]$ instead of summability $[R, \lambda, k]$.

We shall prove the following theorems.

THEOREM A. *If $\sum_{n=0}^{\infty} a_n$ is summable $[R, \lambda, k]$, where $k>0$, to s and $\sum_{n=0}^{\infty} b_n$ is summable $|R, \mu, 0|$ to t, then $\sum_{n=0}^{\infty} c_n$ is summable $[R, \nu, k]$ to sum st.*

THEOREM B. *If $\sum_{n=0}^{\infty} a_n$ is summable $[R, \lambda, k, r]$, where $k>1/r'$ and $r>1$, to s and $\sum_{n=0}^{\infty} b_n$ is summable $|R, \mu, 0|$ to t, then $\sum_{n=0}^{\infty} c_n$ is summable $[R, \nu, k, r]$ to st.*

We observe that Theorem B reduces to Theorem A when $r=1$. It may be mentioned that Theorem A of the present paper includes as a particular case a theorem of Boyd [**1**] for strong Cesàro summability on account of equivalence of summabilities $[R, n, k]$ and $[C, k]$ [**2**].

I am grateful to Dr. (Mrs.) Sulaxana K. Gupta and Dr. Pramila Srivastava for their kind suggestions during the preparation of this paper. I am also indebted to the referee for valuable suggestions according to which I have modified my paper.

4. We require the following lemma.

LEMMA. *Suppose that* $1 \leqq r < \infty$ *and* $k > 0$. *Then, if the series* $\sum_{n=0}^{\infty} a_n$ *is summable* (R, λ, α) *for some* $\alpha > 0$ *to the sum* s, *and*

$$\int_h^X \left| \bar{A}_\lambda^k(u) \right|^r du = o(X^{kr+1}), \quad \text{as} \quad X \to \infty,$$

then it is summable $[R, \lambda, k, r]$ *to the sum* s.

This result is analogous to Flett's Theorem 7 [**4**] on strong Cesàro summability. The lemma follows by combining Corollaries 1 and 2 to Theorem 8 of Srivastava [**5**, p. 66].

5. It is convenient first to prove Theorem B.

Under the hypothesis of the theorem, $\sum_{n=0}^{\infty} a_n$ is summable (R, λ, k) to the sum s and $\sum_{n=0}^{\infty} b_n$ is summable $(R, \mu, 0)$ to the sum t. Applying Theorem 1, we deduce summability $(R, \nu, k+1)$ of $\sum_{n=0}^{\infty} c_n$ to the sum st. Hence by the lemma it is sufficient to prove that

$$(5.1) \qquad \int_h^X \left| \bar{C}_\nu^k(\omega) \right|^r d\omega = o(X^{kr+1}), \quad \text{as} \quad X \to \infty.$$

For $\omega \neq \lambda_p + \mu_q$,

$$\begin{aligned}
\bar{C}_\nu^k(\omega) &= \sum_{\lambda_p + \mu_q < \omega} (\omega - \lambda_p - \mu_q)^{k-1} (\lambda_p + \mu_q) a_p b_q \\
&= \sum_{\mu_q < \omega} \mu_q b_q \sum_{\lambda_p + \mu_q < \omega} (\omega - \lambda_p - \mu_q)^{k-1} a_p \\
&\quad + \sum_{\mu_q < \omega} b_q \sum_{\lambda_p + \mu_q < \omega} (\omega - \lambda_p - \mu_q)^{k-1} \lambda_p a_p \\
&= \sum_{\mu_q < \omega} \mu_q b_q \cdot \frac{1}{(\omega - \mu_q)} A_\lambda^k(\omega - \mu_q) \\
&\quad + \sum_{\mu_q < \omega} \mu_q b_q \cdot \frac{1}{(\omega - \mu_q)} \bar{A}_\lambda^k(\omega - \mu_q) + \sum_{\mu_q < \omega} b_q \bar{A}_\lambda^k(\omega - \mu_q) \\
&= P_1(\omega) + P_2(\omega) + P_3(\omega),
\end{aligned}$$

say. Hence, by Minkowski's inequality, it is enough to prove that, if $P(\omega)$ is any one of $P_1(\omega)$, $P_2(\omega)$, $P_3(\omega)$, then

$$\int_h^X |P(\omega)|^r d\omega = o(X^{kr+1}), \quad \text{as} \quad X \to \infty. \tag{5.2}$$

We observe that

$$|P_3(\omega)|^r = \left| \sum_{\mu_q<\omega} \{(b_q)^{1/r} \overline{A}_\lambda^k(\omega-\mu_q)\} \times \{(b_q)^{1/r'}\} \right|^r.$$

Applying Hölder's inequality for sums with indices r and r', we have

$$|P_3(\omega)|^r \leqq \left\{ \sum_{\mu_q<\omega} |b_q| \, |\overline{A}_\lambda^k(\omega-\mu_q)|^r \right\} \left\{ \sum_{\mu_q<\omega} |b_q| \right\}^{r/r'}. \tag{5.3}$$

We have, since $\sum_{n=0}^{\infty} b_n$ is summable $|R, \mu, 0|$,

$$|P_3(\omega)|^r \leqq M \sum_{\mu_q<\omega} |b_q| \, |\overline{A}_\lambda^k(\omega-\mu_q)|^r,$$

where M is a constant.

Hence

$$\int_h^X |P_3(\omega)|^r d\omega \leqq M \int_h^X \sum_{\mu_q<\omega} |b_q| \, |\overline{A}_\lambda^k(\omega-\mu_q)|^r d\omega.$$

Interchanging the order of integration and summation, we get

$$\begin{aligned} \int_h^X |P_3(\omega)|^r d\omega &\leqq M \sum_{\mu_q<X} |b_q| \int_{\mu_q}^X |\overline{A}_\lambda^k(\omega-\mu_q)|^r d\omega \\ &= M \sum_{\mu_q<X} |b_q| \, o(X^{kr+1}) \\ &= o(X^{kr+1}), \end{aligned}$$

by virtue of the hypothesis.

We further observe that

$$|P_1(\omega)|^r = \left| \sum_{\mu_q<\omega} \left\{ (\mu_q b_q)^{1/r} \frac{A_\lambda^k(\omega-\mu_q)}{(\omega-\mu_q)} \right\} \times \{(\mu_q b_q)^{1/r'}\} \right|^r.$$

Applying Hölder's inequality for sums with indices r and r', we have

$$(5.4)\quad | P_1(\omega) |^r \leqq \left\{ \sum_{\mu_q<\omega} | \mu_q b_q | \frac{| A_\lambda^k(\omega - \mu_q) |^r}{(\omega - \mu_q)^r} \right\} \left\{ \sum_{\mu_q<\omega} | \mu_q b_q | \right\}^{r/r'}$$

$$= o(\omega^{r-1}) \sum_{\mu_q<\omega} | \mu_q b_q | \frac{| A_\lambda^k(\omega - \mu_q) |^r}{(\omega - \mu_q)^r}.$$

Hence, by using Theorem 1 of Srivastava [5] and the hypothesis of the theorem,

$$\int_h^X | P_1(\omega) |^r d\omega \leqq \sum_{\mu_q<X} | \mu_q b_q | \int_{\mu_q}^X o(\omega^{r-1}) \frac{| A_\lambda^k(\omega - \mu_q) |^r}{(\omega - \mu_q)^r} d\omega$$

$$\leqq o(X^{r-1}) \sum_{\mu_q<X} | \mu_q b_q | \int_{\mu_q}^X o(1)(\omega - \mu_q)^{r(k-1)} d\omega$$

$$= o(X^{r-1}) \sum_{\mu_q<X} | \mu_q b_q | \, o(X^{r(k-1)+1})$$

$$= o(X^{r-1}) o(X^{r(k-1)+1}) \sum_{\mu_q<X} | \mu_q b_q |$$

$$= o(X^{kr+1}),$$

provided $k > 1/r'$.

Similarly we can prove that

$$\int_h^X | P_2(\omega) |^r d\omega = o(X^{kr+1}), \quad \text{as} \quad X \to \infty.$$

Thus collecting our results, we have

$$\int_h^X | \overline{C}_\nu^k(\omega) |^r d\omega = o(X^{kr+1}), \quad \text{as} \quad X \to \infty.$$

This completes the proof of Theorem B.

6. Proof of Theorem A. The proof of this theorem follows immediately from Theorem B by omitting the last factors in (5.3) and (5.4).

References

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ON HYPONORMAL OPERATORS

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1. An operator T defined on a Hilbert space H is said to be hyponormal if $T^*T-TT^*\geqq 0$, or equivalently if $\|T^*x\|\leqq\|Tx\|$ for every $x\in H$. An operator T is said to be seminormal if either T or T^* is hyponormal. If T is hyponormal, then $T-zI$ is also hyponormal for all complex values of z.

The spectrum of an operator T, in symbols $\sigma(T)$, is the set of all those complex numbers z for which $T-zI$ is not invertible. A complex number z is said to be an approximate proper value for the operator T in case there exists a sequence x_n such that $\|x_n\|=1$ and $\|(T-zI)x_n\|\to 0$. The approximate point spectrum of an operator T, in symbols $\Pi(T)$, is the set of approximate proper values of T. The numerical range of an operator T, denoted by $W(T)$, is the set defined by the relation

$$W(T) = \{(Tx, x): \|x\| = 1\}.$$

Cl $(W(T))$ will, as usual, denote the closure of $W(T)$. An operator S is said to be similar to an operator T in case there exists an invertible operator A such that $S=A^{-1}TA$.

In this note, all the operators will relate to a Hilbert space H.

We shall prove the following theorem.

THEOREM. *Let N be a hyponormal operator. If for an arbitrary operator A, for which $0\notin$ Cl $(W(A))$, $AN=N^*A$, then N is self-adjoint.*

Received by the editors April 11, 1966.