INVARIANCE OF THE HOMOLOGY OF A LATTICE[1]

JOHN MATHER

G.-C. Rota, D. Kan, F. Peterson, and G. Whitehead have constructed a homology theory for finite lattices. Their definition is given in terms of a choice of "cross-cut" for the given lattice; however it was conjectured by G.-C. Rota [**4**, p. 356] that the Betti numbers so obtained are independent of the choice of cross-cut. It is an immediate consequence of the theorem proved in this paper that this is so. Jon Folkman [**2**] has given an independent proof of the invariance of the homology of a lattice. He shows that the homology of any cross-cut is the same as the homology of the complex whose vertices are the elements of the lattice other than 0 or 1 and whose simplices are the totally ordered subsets of the vertices. His proof is valid for infinite as well as finite lattices.

As in [**4**], a *cross-cut* of a finite lattice is a subset C of L such that:

(a) C contains neither 0 nor 1,

(b) no two elements of C are comparable,

(c) every chain stretched from 0 to 1 meets C.

A *spanning subset* of L is a subset whose join is 1 and whose meet is 0. Given a cross-cut C of L, we define a simplicial complex $K(C, L)$. The vertices of $K(C, L)$ are the elements of C. The simplices of $K(C, L)$ are the subsets of C which do not span L.

THEOREM. *If C and C' are two cross-cuts of L, then $K(C, L)$ and $K(C', L)$ have the same homotopy type.*

Let U_L (resp. V_L) denote the cross-cut of L whose elements are the maximal (resp. minimal) elements in $L-\{0, 1\}$. If C is any cross-cut of L, we denote the first barycentric subdivision of $K(C, L)$ by $K'(C, L)$, and define a map $f_{C,L}$ from the set of vertices of $K'(C, L)$ to the set of vertices of vertices of $K'(U_L, L)$ as follows. If S is a vertex of $K'(C, L)$ (in other words, a simplex of $K(C, L)$, i.e., a subset of C which does not span L) then

$$\begin{aligned} f_{C,L}(S) &= \{x \in U_L \colon x \geqq \bigvee S\} \qquad \text{if } \bigvee S \neq 1, \\ &= \{x \in U_L \colon x \geqq \bigwedge S\} \qquad \text{if } \bigvee S = 1. \end{aligned}$$

To show that $f_{C,L}(S)$ is really a vertex of $K'(U_L, L)$, we must show that it does not span L. If $\bigvee S \neq 1$ then $\bigwedge f_{C,L}(S) > \bigvee S > 0$. If $\bigvee S = 1$

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then, since S does not span, $\wedge S>0$. Hence, $\wedge f_{C,L}(S)\geqq\wedge S>0$.

We will prove our theorem by showing that $f_{C,L}$ is a simplicial map and that its geometric realization (which will also be denoted by $f_{C,L}$) is a homotopy equivalence.

LEMMA 1. *$f_{C,L}$ is a simplicial map.*

PROOF. It suffices to show that if S and T are simplices of $K(C, L)$ with $S\subseteq T$, then either $f_{C,L}(S)\subseteq f_{C,L}(T)$ or $f_{C,L}(T)\subseteq f_{C,L}(S)$. There are two cases. If $\vee T<1$, then $\vee S<1$ and it follows from $\vee S\leqq\vee T$ that $f_{C,L}(T)\subseteq f_{C,L}(S)$. If $\vee T=1$ then $x\in f_{C,L}(S)$ implies $x\geqq\wedge S\geqq\wedge T$ and hence $x\in f_{C,L}(T)$. Hence $f_{C,L}(S)\subseteq f_{C,L}(T)$ in this case.

The proof that $f_{C,L}$ is a homotopy equivalence will be by induction on

$$p(C, L) = \operatorname{card}\{x \in L\colon \exists y \in C,\ x < y\}.$$

Basis for the induction: $p(C, L)=1$. (This is a special case of Theorem 1 of [1].) In this case $C= V_L$, and the map $f=f_{C,L}$ is given by

$$f(S) = \{x \in U_L\colon x \geqq \vee S\}$$

for any vertex S of $K'(V_L, L)$. Dually there is a simplicial map

$$g\colon K'(U_L, L) \to K'(V_L, L)$$

given by

$$g(S) = \{x \in V_L\colon x \leqq \wedge S\}$$

for any vertex S of $K'(U_L, L)$.

Observe that there is a partial order ($\leqq$) on the vertices of the barycentric subdivision K' of any simplicial complex K, induced by the inclusion relation between simplices of K via the correspondence between simplices of K and vertices of K'. In this notation it is easily seen that if S and T are vertices of $K'(U_L, L)$ and $S\leqq T$ then

$$S \leqq fg(S) \leqq fg(T). \tag{1}$$

We define a homotopy

$$h\colon |K'(U_L, L)| \times I \to |K(U_L, L)|$$

by setting

$$h(x, t) = t\cdot fg(x) + (1 - t)\cdot x, \qquad x \in |K'(U_L, L)|,\ t \in I.$$

Clearly if the definition of h is meaningful, then h is a homotopy connecting fg and the identity. In order to show that the definition makes sense, we must show that for any $x\in|K'|$ there is a simplex in K

whose geometric realization contains both x and $fg(x)$. To see this, let $\sigma=\langle V_1, \cdots, V_n\rangle$ be a simplex of K' whose geometric realization contains x. Suppose the V's are ordered so that $V_1\leqq V_2\leqq \cdots \leqq V_n$. Let τ be the simplex of K corresponding to $fg(V_n)$. Since $V_i\leqq fg(V_n)$ and $fg(V_i)\leqq fg(V_n)$, by (1), $V_i\in|\tau|$ and $fg(V_i)\in|\tau|$ for $i=1, 2, \cdots, n$. Hence, $|\sigma|\subseteq|\tau|$ and since f, as a map from $|K'|$ to $|K|$, is linear, $fg[|\sigma|]\subseteq|\tau|$. In particular $x\in|\tau|$ and $fg(x)\in|\tau|$. Hence, $fg\sim 1$. Similarly $gf\sim 1$.

Inductive step: $p(C, L)>1$. The proof of the inductive step is based upon the following lemma which will be proved at the end of the paper.

LEMMA 2. *Let $X\cup Y$ and $Z\cup W$ be simplicial complexes such that X, Y, Z, W are subcomplexes. Let F: $X\cup Y\to Z\cup W$ be a simplicial map such that*

$$F[X]\subseteq Z, \qquad F[Y]\subseteq W, \tag{2}$$

and also such that

$$F_1 = F\,|\,X\colon X\to Z, \tag{3}$$

$$F_2 = F\,|\,Y\colon Y\to W, \tag{4}$$

$$F_3 = F\,|\,X\cap Y\colon X\cap Y\to Z\cap W, \tag{5}$$

are homotopy equivalences. Then F is a homotopy equivalence.

Since $p(C, L)>1$ there is an $x\in V_L-C$. Let

$$L_1 = L - \{x\},$$
$$L_2 = \{y\in L\colon y\geqq x\}.$$

The partial order on L induces partial orders on L_1 and L_2 in which L_1 and L_2 are lattices. The zero of L_2 is x. Let $C_2=C\cap L_2$. Let σ be the closed simplex on C_2, i.e., the simplicial complex whose vertices are the members of C_2 and whose simplices are all subsets of C_2. Similarly let τ be the closed simplex on U_{L_2}. Then $\sigma\subseteq K(C, L)$ and $\tau\subseteq K(U_{L_2}, L)$. It is easily verified that

$$K(C, L) = K(C, L_1)\cup\sigma, \tag{6}$$

$$K(U_L, L) = K(U_L, L_1)\cup\tau, \tag{7}$$

$$K(C_2, L_2) = K(C, L_1)\cap\sigma, \tag{8}$$

$$K(U_{L_2}, L_2) = K(U_L, L_1)\cap\tau, \tag{9}$$

and

$$(10)\qquad f_{C,L}[\,|K(C, L_1)|\,] = f_{C,L}[\,|K'(C, L_1)|\,] \subseteq |K'(U_L, L_1)| = |K(U_L, L_1)|,$$

$$(11)\qquad f_{C,L}[\,|\sigma|\,] = f_{C,L}[\,|\sigma'|\,] \subseteq |\tau'| = |\tau|,$$

$$(12)\qquad f_{C,L}\Big|\,|K(C, L_1)| = f_{C,L_1},$$

$$(13)\qquad f_{C,L}\Big|\,|K(C, L_1)| \cap |\sigma| = f_{C_2,L_2}.$$

Set $|K(C, L_1)| = X$, $|\sigma| = Y$, $|K(U_L, L_1)| = Z$ and $|\tau| = W$. Let $F = f_{C,L}$. Then F maps $X \cup Y$ to $Z \cup W$, by (6) and (7), and F, X, Y, Z, W satisfy (2), by (10) and (11). Since $p(C, L_1) < p(C, L)$ and $p(C_2, L_2) < p(C, L)$ it follows from the inductive hypothesis that f_{C,L_1} and f_{C_2,L_2} are homotopy equivalences. Hence by (12), F_1 (defined by (3)) is a homotopy equivalence. Also by (8), (9), and (13), F_3 (defined by (5)) is a homotopy equivalence. Finally $Y = |\sigma|$ and $W = |\tau|$ are contractible, so F_2 is a homotopy equivalence. Hence, by Lemma 3, $F = f_{C,L}$ is a homotopy equivalence.

PROOF OF LEMMA 2. Let $G: P \to Q$ be any continuous map. Let $M(G)$ denote the mapping cylinder of G, namely $(P \times I) \cup Q/E$ where E is the equivalence relation generated by the relations $\langle x, 1\rangle E G(x)$ for all $x \in P$. To prove Lemma 2 we will need the following lemma.

LEMMA 3. *If $G: P \to Q$ is a cellular map of* CW *complexes then G is a homotopy equivalence if and only if $P \times \{0\}$ is a deformation retract of $M(G)$.*

To say $P \times \{0\}$ is a deformation retract of $M(G)$ is to say that there is a deformation retraction of $M(G)$ on $P \times \{0\}$, i.e., a map

$$H: M(G) \times I \to M(G)$$

such that

$$\begin{aligned} H(x, 0) &= x, && x \in M(G),\\ H(x, t) &= x, && x \in P, \quad t \in I,\\ H(x, 1) &\in P, && x \in M(G). \end{aligned}$$

The "if" part of Lemma 3 is straightforward. The "only if" part follows from the fact that if G is a homotopy equivalence then $\pi_r(M(G), P \times \{0\}) = 0$, $r = 1, 2, \cdots$, by the homotopy exact sequence, and from Theorem 1.7 of Chapter VII of [2] and the comment following it.

Returning to the proof of Lemma 2, it follows from Lemma 3 that there is a deformation retraction H_3 of $M(F_3)$ on $(X \cap Y) \times \{0\}$. By two applications of the homotopy extension property of CW complexes it follows that there is an extension

$$H_3' ; M(F) \times I \to M(F)$$

of H_3 such that

$$\begin{aligned} &H_3'(x, 0) = x, && x \in M(F), \\ &H_3'(x, t) = x, && x \in (X \cup Y) \times \{0\}, \quad t \in I, \\ &H_3'(M(F_1) \times I) \subseteq M(F_1), && \\ &H_3'(M(F_2) \times I) \subseteq M(F_2). && \end{aligned}$$

By Lemma 3 there are deformation retractions H_1 of $M(F_1)$ on $X\times\{0\}$ and H_2 of $M(F_2)$ on $Y\times\{0\}$. We define a homotopy H: $M(F)\times I\to M(F)$ by

$$\begin{aligned} &H(x, t) = H_3'(x, 2t), && x \in M(F), && 0 \leqq t \leqq 1/2, \\ &H(x, t) = H_1(H_3'(x, 1), 2t - 1), && x \in M(F_1), && 1/2 \leqq t \leqq 1, \\ &H(x, t) = H_2(H_3'(x, 1), 2t - 1), && x \in M(F_2), && 1/2 \leqq t \leqq 1. \end{aligned}$$

It is easily verified that H is well defined and is a deformation retraction on $(X\cup Y)\times\{0\}$. Hence, by Lemma 3, H is a homotopy equivalence.

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Princeton University