ON THE EXISTENCE AND UNIQUENESS OF THE REAL LOGARITHM OF A MATRIX

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1. **Introduction.** Consider the exponential matrix equation

$$C = e^X, \tag{1.1}$$

where C is a given real matrix of dimension $n \times n$. What we shall examine in this paper are the conditions under which a *real* matrix X exists to satisfy (1.1) and, obtaining existence, the conditions under which such a solution is unique.

The significance of this study can derive from a number of sources, one of which is the mathematical modeling of dynamic systems [**1**].

2. **A sketch of the results.** According to Gantmacher [**2**, pp. 239–241], the solution to (1.1) proceeds in the following way:

We reduce C to its Jordan *normal form* J via the similarity transformation

$$S^{-1}CS = J, \tag{2.1}$$

whereby (1.1) becomes

$$J = S^{-1}e^X S = \exp(S^{-1} X S). \tag{2.2}$$

We then take the natural logarithm of both sides of (2.2) and invert the similarity transformation to obtain the desired solution(s) X.

As we will show rigorously, a *real* solution exists provided C is nonsingular and each elementary divisor (Jordan block) of C corresponding to a negative eigenvalue occurs an even number of times. This assures that the complex part of X will have complex conjugate elementary divisors (Jordan blocks).

The possible nonuniqueness of the solution can arise in two ways as we will demonstrate: (1) because the matrix C has complex eigenvalues and hence provides log J with at least a countable infinity of periodic values, and (2) because the similarity transformation which relates J to C uniquely[2] via (2.1) may not relate log J to X uniquely via (2.2), in which case an uncountable infinity of solutions results.

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[2] A Jordan form J is unique to within an ordering of diagonal blocks.

Case (2) corresponds to the situation where log J cannot be expressed as a power series in J.

3. **The mathematical preliminaries.** It is well known that the matrix S in (2.1) is not unique, although J is uniquely related to C. In this regard, the following lemma is of interest.

LEMMA 1. *Every matrix S which takes a given matrix C into its Jordan form J via the relation*

$$C = SJS^{-1}, \tag{3.1}$$

differs from any other matrix $\tilde{S}$ which does the same thing, i.e.,

$$C = \tilde{S}J\tilde{S}^{-1} \tag{3.2}$$

only by a multiplicative nonsingular matrix factor K which is one of a continuum of such matrices that commute with J and provide the identity

$$\tilde{S} = SK. \tag{3.3}$$

PROOF. Equate (3.1) to (3.2) and rearrange terms to obtain

$$(S^{-1}\tilde{S})J = J(S^{-1}\tilde{S}).$$

From this it is obvious that $S^{-1}\tilde{S}$ must be a matrix, say K, which commutes with J, and is nonsingular, wherefrom (3.3) follows directly to complete the proof of the lemma.

Clearly, now, if S is replaced by the more general transformation $\tilde{S}=SK$, equations (2.1) and (2.2) remain exactly the same, since every K commutes with J. However, after the logarithm of J is taken, K may not commute with log J, so that for complete generality we must write

$$X = SK(\log J)K^{-1}S^{-1}. \tag{3.4}$$

The logarithm of J is well defined [2, p. 100] in terms of its real Jordan blocks $J_1, \cdots, J_m$, $m \leqq n$:

$$\log J = \operatorname{diag}\{\log J_1, \cdots, \log J_m\}. \tag{3.5}$$

Typically, if the kth block is of dimension $(\alpha_k+1)\times(\alpha_k+1)$ and corresponds to the real elementary divisor

$$(\lambda - \lambda_k)^{\alpha_k+1}, \tag{3.6}$$

where λ_k is a real eigenvalue of C not necessarily different from λ_h $(h \neq k)$, then

$$
J_k = \begin{bmatrix} \lambda_k & 1 & \cdots & 0 \\ & \lambda_k & \ddots & \\ & & \ddots & 1 \\ 0 & \cdots & \cdots & \lambda_k \end{bmatrix} \tag{3.7}
$$

and

$$
\log J_k = \begin{bmatrix} \log \lambda_k & 1/\lambda_k & \cdots & -((-\lambda_k)^{-\alpha_k})/\alpha_k \\ & \log \lambda_k & \ddots & \\ & & \ddots & 1/\lambda_k \\ 0 & \cdots & \cdots & \log \lambda_k \end{bmatrix}. \tag{3.8}
$$

If, on the other hand, the kth block corresponds to the complex conjugate elementary divisors

$$
(\lambda - \lambda_k)^{\beta_k+1} \quad \text{and} \quad (\lambda - \lambda_k^*)^{\beta_k+1}, \tag{3.9}
$$

where $\lambda_k = u_k + iv_k$ is a complex eigenvalue of C and λ_k^* is its complex conjugate, then the block dimensions are $2(\beta_k+1)\times 2(\beta_k+1)$ and

$$
J_k = \begin{bmatrix} L_k & I & \cdots & 0 \\ & L_k & \ddots & \\ & & \ddots & I \\ 0 & & & L_k \end{bmatrix}, \tag{3.10}
$$

where

$$
L_k = \begin{bmatrix} u_k & -v_k \\ v_k & u_k \end{bmatrix}. \tag{3.11}
$$

For this complex case,

$$
\log J_k = \begin{bmatrix} \log L_k & L_k^{-1} & \cdots & -((-L_k^{-1})^{\beta_k})/\beta_k \\ & \log L_k & \ddots & \\ & & \ddots & L_k^{-1} \\ 0 & \cdots & \cdots & \log L_k \end{bmatrix}. \tag{3.12}
$$

Since all matrix logarithms are defined ultimately by the matrix exponential, e.g.,

$$
J_k = \exp(\log J_k),
$$

it follows that such logarithms are multivalued functions of the type

$$
\log J_k = \mathrm{LOG}\, J_k + D, \tag{3.13}
$$

where LOG is the principal value and D is one of an infinity of matrices that commute with LOG J_k and satisfy the relation $e^D = I$.

The nature of D depends on whether the λ_k belonging to J_k is real or complex. If λ_k is *real*, the eigenvalues of log J_k are its diagonal elements, and from a theorem in Gantmacher [2, p. 158], these must be equal. Thus

$$\log J_k = \text{LOG}\, J_k + i2\pi q_k I, \quad \lambda_k \text{ real}, \tag{3.14}$$

where $q_k = 0, \pm 1, \pm 2, \cdots$.

On the other hand, if Im $\lambda_k \neq 0$, the real and imaginary parts of the eigenvalues of log J_k appear respectively on the main and skew diagonals of the 2×2 diagonal blocks of log J_k. Again Gantmacher's theorem can be used, this time to infer that the diagonal blocks of log J_k must be equal. Thus

$$\log J_k = \text{LOG}\, J_k + 2\pi(iq_k I + r_k E), \quad \text{Im}\, \lambda_k \neq 0, \tag{3.15}$$

where both q_k and r_k can assume the values $0, \pm 1, \pm 2, \cdots$, and where

$$E = \text{diag}\left\{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \cdots, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\right\}.$$

4. **Development of results.** From expressions (3.14) and (3.15) we can see that if no constraints are put on the solution $X = SK(\text{diag}\{\log J_1, \cdots, \log J_m\})K^{-1}S^{-1}$, then at least a countable infinity of X's are produced. In this paper we apply, for physical reasons [1], the constraint that X be real, the immediate consequence of which is that the complex elementary divisors (Jordan blocks) of X must appear in complex conjugate pairs. The question of existence under this constraint is answered by the following theorem.

THEOREM 1. *Let C be a real square matrix. Then there exists a real solution X to the equation $C = e^X$ if and only if* (*) *C is nonsingular and each elementary divisor* (*Jordan block*) *of C belonging to a negative eigenvalue occurs an even number of times.*

PROOF.[3] (i) *Necessity.* Let X be real such that $C = e^X$. If any complex eigenvalues of X exist, they must correspond to complex conjugate elementary divisors. Hence, we may suppose that the elementary divisors of X are

$$\begin{aligned} &(\lambda - z_k)^{a_k}, \quad z_k \text{ real}, \\ &(\lambda - z_k)^{b_k} \text{ and } (\lambda - \overset{*}{z_k})^{b_k}, \quad \text{Im}\, z_k \neq 0. \end{aligned} \tag{4.1}$$

[3] The proof in this form is due essentially to the reviewer of the paper.

Since $de^{\lambda}/d\lambda \neq 0$ for all finite λ, it follows from a theorem in Gantmacher [2, p. 158] that the elementary divisors of $C = e^X$ are

$$(4.2) \qquad \begin{array}{l} (\lambda - e^{z_k})^{a_k}, \quad z_k \text{ real} \\ (\lambda - e^{z_k})^{b_k} \text{ and } (\lambda - e^{z_k^*})^{b_k}, \quad \operatorname{Im} z_k \neq 0. \end{array}$$

In no event is $e^{z_k} = 0$. Moreover, $e^{z_k} < 0$ only if $\operatorname{Im} z_k \neq 0$, in which case $e^{z_k} = e^{z_k^*}$. Thus the negative eigenvalues of C must associate with elementary divisors which occur in pairs. Hence C must satisfy (*).

(ii) *Sufficiency*. Conversely, let C satisfy (*). Its eigenvalues λ_k are as specified by (3.6) or (3.9). For those λ_k that are real and negative we can write $\lambda_k = e^{z_k} = e^{z_k^*}$, where $z_k = \operatorname{LOG}|\lambda_k| + i\pi$. Moreover, by the last part of (*), the corresponding elementary divisors are $(\lambda - e^{z_k})^{\alpha_k+1}$ and $(\lambda - e^{z_k^*})^{\alpha_k+1}$. Since, also, C is real, we may suppose that all the elementary divisors of C are given by (4.2). Consider, now, the class of matrices with elementary divisors (4.1). Clearly there exists some real matrix Y in this class. By the theorem quoted from Gantmacher, the function e^Y must be similar to C, so that a real matrix T can be found such that

$$C = T^{-1}e^Y T = \exp(T^{-1}YT).$$

Identify X with $T^{-1}YT$ to confirm the sufficiency of (*).

THEOREM 2. *Let C be a real square matrix. Then the equation $C = e^X$ has a unique real solution X if and only if* (**) *all the eigenvalues of C are positive real and no elementary divisor* (*Jordan block*) *of C belonging to any eigenvalue appears more than once.*

PROOF. (i) *Sufficiency*. All the solutions to $C = e^X$ are given by (3.4), (3.5):

$$X = SK(\operatorname{diag}\{\log J_1, \cdots, \log J_m\})K^{-1}S^{-1},$$

where $\log J_k$ is given by (3.14) or (3.15). Clearly, if (**) holds, $\operatorname{LOG} J_k$ is real, whereas $\log J_k = \operatorname{LOG} J_k + i2\pi q_k I$ is complex and has no complex conjugate in the set $\log J_h = \operatorname{LOG} J_h + i2\pi q_h I$, $h \neq k$. Hence, for every k the parameter q_k must be zero, and for every set of blocks (say $J_k, J_{k+1}, \cdots, J_{k+\gamma_k}$) which belongs to the eigenvalue λ_k there exists the unique set $\operatorname{LOG} J_k, \operatorname{LOG} J_{k+1}, \cdots, \operatorname{LOG} J_{k+\gamma_k}$ which belongs to the eigenvalue $\operatorname{LOG} \lambda_k$. Hence [2, p. 220] every K that commutes with J must also commute with $\log J$ in (3.4), and (**) is sufficient for X to be real and unique.

(ii) *Necessity*. Take the contradictions to (**) which satisfy condition (*) of Theorem 1. For example, assume C to have positive real eigenvalues which belong to Jordan blocks that appear more than once, or assume C to have negative real eigenvalues (whose blocks must occur in pairs), or assume C to have complex conjugate eigenvalues.

Suppose, first, that λ_k is real and corresponds to the identical blocks J_k, J_{k+1}. If in (3.14) we choose $q_k = -q_{k+1}$ for λ_k positive real and $q_k = -(1+q_{k+1})$ for λ_k negative real, we obtain the complex conjugate blocks $\log J_k$, $\log J_{k+1}$. Hence a continuum set of K matrices that commuted with J will not commute with $\log J$, and a continuum of real X's will arise from (3.4).

Suppose now, that some pair of eigenvalues of C are complex conjugate. If they correspond to Jordan blocks that appear more than once (say J_k, J_{k+1}), then by taking $q_k = -q_{k+1}$ in (3.15) we obtain a continuum set of X's from (3.4). If the blocks appear only once, q_k must be zero for all k or else $\log J_k$ will be a complex block without a conjugate. However, r_k in (3.15) can be any integer. If any two blocks (say J_k, J_{k+1}) are *not* identical but belong to the same eigenvalue λ_k, the fact that r_k need not equal r_{k+1} makes it possible for $\log J_k$ and $\log J_{k+1}$ to belong to different eigenvalues. Hence not every K will commute with $\log J$ and again a continuum of X's result. Finally, if no two blocks of J belong to the same complex eigenvalue, every K that commutes with J will also commute with $\log J$, provided the Jordan blocks for the real eigenvalues appear only once. But r_k can still be any integer, which leads to a countable infinity of $\log J$'s, and hence to a countable infinity of X's. Thus (**) is necessary.

COROLLARY. *Let C be a real square matrix and let $C = e^X$ have more than one real solution X. Then there exists an infinity of real solutions X which are*

(a) Countable *if all real eigenvalues of C are positive such that their Jordan blocks appear only once and C has complex eigenvalues none of which belongs to more than one Jordan block*;

(b) Uncountable *if any real eigenvalues of C are negative, or if any positive real eigenvalues belong to Jordan blocks that appear more than once, or if any complex conjugate eigenvalues belong to more than one Jordan block.*

REFERENCES

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