ON THE EXISTENCE AND UNIQUENESS OF THE 
REAL LOGARITHM OF A MATRIX

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1. Introduction. Consider the exponential matrix equation

\[ C = e^{X}, \]

where \( C \) is a given real matrix of dimension \( n \times n \). What we shall examine in this paper are the conditions under which a real matrix \( X \) exists to satisfy (1.1) and, obtaining existence, the conditions under which such a solution is unique.

The significance of this study can derive from a number of sources, one of which is the mathematical modeling of dynamic systems [1].

2. A sketch of the results. According to Gantmacher [2, pp. 239-241], the solution to (1.1) proceeds in the following way:

We reduce \( C \) to its Jordan normal form \( J \) via the similarity transformation

\[ S^{-1}CS = J, \]

whereby (1.1) becomes

\[ J = S^{-1}e^{X}S = \exp(S^{-1}X S). \]

We then take the natural logarithm of both sides of (2.2) and invert the similarity transformation to obtain the desired solution(s) \( X \).

As we will show rigorously, a real solution exists provided \( C \) is nonsingular and each elementary divisor (Jordan block) of \( C \) corresponding to a negative eigenvalue occurs an even number of times. This assures that the complex part of \( X \) will have complex conjugate elementary divisors (Jordan blocks).

The possible nonuniqueness of the solution can arise in two ways as we will demonstrate: (1) because the matrix \( C \) has complex eigenvalues and hence provides \( \log J \) with at least a countable infinity of periodic values, and (2) because the similarity transformation which relates \( J \) to \( C \) uniquely via (2.1) may not relate \( \log J \) to \( X \) uniquely via (2.2), in which case an uncountable infinity of solutions results.

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2 A Jordan form \( J \) is unique to within an ordering of diagonal blocks.
Case (2) corresponds to the situation where \( \log J \) cannot be expressed as a power series in \( J \).

3. The mathematical preliminaries. It is well known that the matrix \( S \) in (2.1) is not unique, although \( J \) is uniquely related to \( C \). In this regard, the following lemma is of interest.

**Lemma 1.** Every matrix \( S \) which takes a given matrix \( C \) into its Jordan form \( J \) via the relation

\[
C = SJS^{-1},
\]

differs from any other matrix \( \tilde{S} \) which does the same thing, i.e.,

\[
C = \tilde{S}J\tilde{S}^{-1}
\]

only by a multiplicative nonsingular matrix factor \( K \) which is one of a continuum of such matrices that commute with \( J \) and provide the identity

\[
\tilde{S} = SK.
\]

**Proof.** Equate (3.1) to (3.2) and rearrange terms to obtain

\[
(S^{-1}\tilde{S})J = J(S^{-1}\tilde{S}).
\]

From this it is obvious that \( S^{-1}\tilde{S} \) must be a matrix, say \( K \), which commutes with \( J \), and is nonsingular, wherefrom (3.3) follows directly to complete the proof of the lemma.

Clearly, now, if \( S \) is replaced by the more general transformation \( \tilde{S} = SK \), equations (2.1) and (2.2) remain exactly the same, since every \( K \) commutes with \( J \). However, after the logarithm of \( J \) is taken, \( K \) may not commute with \( \log J \), so that for complete generality we must write

\[
X = SK(\log J)K^{-1}S^{-1}.
\]

The logarithm of \( J \) is well defined [2, p. 100] in terms of its real Jordan blocks \( J_1, \cdots, J_m, m \leq n \):

\[
\log J = \text{diag}\{ \log J_1, \cdots, \log J_m \}.
\]

Typically, if the \( k \)th block is of dimension \( (\alpha_k+1) \times (\alpha_k+1) \) and corresponds to the real elementary divisor

\[
(\lambda - \lambda_k)\alpha_k+1,
\]

where \( \lambda_k \) is a real eigenvalue of \( C \) not necessarily different from \( \lambda_h \) \((h \neq k)\), then
(3.7) \[ J_k = \begin{bmatrix} \lambda_k & 1 & \cdots & 0 \\ \lambda_k & & & \\ & \ddots & & \\ 0 & \cdots & \cdots & \lambda_k \end{bmatrix} \]

and

(3.8) \[ \log J_k = \begin{bmatrix} \log \lambda_k & 1/\lambda_k & \cdots & -((-\lambda_k^{-\alpha_k})/\alpha_k) \\ \log \lambda_k & & & \\ & \ddots & & \\ 0 & \cdots & \cdots & \log \lambda_k \end{bmatrix} \]

If, on the other hand, the \( k \)th block corresponds to the complex conjugate elementary divisors

(3.9) \[ (\lambda - \lambda_k)^{\beta_k+1} \quad \text{and} \quad (\lambda - \lambda_k^*)^{\beta_k+1}, \]

where \( \lambda_k = u_k + iv_k \) is a complex eigenvalue of \( C \) and \( \lambda_k^* \) is its complex conjugate, then the block dimensions are \( 2(\beta_k+1) \times 2(\beta_k+1) \) and

(3.10) \[ J_k = \begin{bmatrix} L_k & I & \cdots & 0 \\ & L_k & \cdots & \\ & & \ddots & I \\ 0 & \cdots & \cdots & L_k \end{bmatrix}, \]

where

(3.11) \[ L_k = \begin{bmatrix} u_k & -v_k \\ v_k & u_k \end{bmatrix}. \]

For this complex case,

(3.12) \[ \log J_k = \begin{bmatrix} \log L_k & L_k^{-1} & \cdots & -((-L_k^{-1})^{\beta_k})/eta_k \\ \log L_k & \cdots & & \\ & \ddots & \cdots & L_k^{-1} \\ 0 & \cdots & \cdots & \log L_k \end{bmatrix}. \]

Since all matrix logarithms are defined ultimately by the matrix exponential, e.g.,

\[ J_k = \exp(\log J_k), \]

it follows that such logarithms are multivalued functions of the type

(3.13) \[ \log J_k = \text{LOG} \, J_k + D, \]
where \( \text{LOG} \) is the principal value and \( D \) is one of an infinity of matrices that commute with \( \text{LOG} \) \( J_k \) and satisfy the relation \( e^D = I \).

The nature of \( D \) depends on whether the \( \lambda_k \) belonging to \( J_k \) is real or complex. If \( \lambda_k \) is real, the eigenvalues of \( \log J_k \) are its diagonal elements, and from a theorem in Gantmacher [2, p. 158], these must be equal. Thus

\[
(3.14) \quad \log J_k = \text{LOG} J_k + i2\pi q_k I, \quad \lambda_k \text{ real},
\]

where \( q_k = 0, \pm 1, \pm 2, \cdots \).

On the other hand, if \( \text{Im} \lambda_k \neq 0 \), the real and imaginary parts of the eigenvalues of \( \log J_k \) appear respectively on the main and skew diagonals of the \( 2 \times 2 \) diagonal blocks of \( \log J_k \). Again Gantmacher's theorem can be used, this time to infer that the diagonal blocks of \( \log J_k \) must be equal. Thus

\[
(3.15) \quad \log J_k = \text{LOG} J_k + 2\pi(iq_k I + r_k E), \quad \text{Im} \lambda_k \neq 0,
\]

where both \( q_k \) and \( r_k \) can assume the values \( 0, \pm 1, \pm 2, \cdots \), and where

\[
E = \text{diag}\left\{\begin{bmatrix} 0 & -1 \\ 1 & \phantom{-}0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 & -1 \\ 1 & \phantom{-}0 \end{bmatrix}\right\}.
\]

4. Development of results. From expressions (3.14) and (3.15) we can see that if no constraints are put on the solution \( X = SK(\text{diag}\{\log J_1, \ldots, \log J_m\})K^{-1}S^{-1} \), then at least a countable infinity of \( X \)'s are produced. In this paper we apply, for physical reasons [1], the constraint that \( X \) be real, the immediate consequence of which is that the complex elementary divisors (Jordan blocks) of \( X \) must appear in complex conjugate pairs. The question of existence under this constraint is answered by the following theorem.

**Theorem 1.** Let \( C \) be a real square matrix. Then there exists a real solution \( X \) to the equation \( C = e^X \) if and only if (*) \( C \) is nonsingular and each elementary divisor (Jordan block) of \( C \) belonging to a negative eigenvalue occurs an even number of times.

**Proof.** (i) **Necessity.** Let \( X \) be real such that \( C = e^X \). If any complex eigenvalues of \( X \) exist, they must correspond to complex conjugate elementary divisors. Hence, we may suppose that the elementary divisors of \( X \) are

\[
(4.1) \quad (\lambda - z_k)^a_k, \quad z_k \text{ real}, \quad (\lambda - z_k^*)^{b_k} \text{ and } (\lambda - z_k^{*})^{b_k}, \quad \text{Im} \; z_k \neq 0.
\]

* The proof in this form is due essentially to the reviewer of the paper.
Since \( \frac{d e^X}{d \lambda} \neq 0 \) for all finite \( \lambda \), it follows from a theorem in Gantmacher [2, p. 158] that the elementary divisors of \( C = e^X \) are

\[
(\lambda - e^{z_k})^{a_k}, \quad z_k \text{ real}
\]

\[
(\lambda - e^{z_k})^{b_k} \quad \text{and} \quad (\lambda - e^{z_k})^{b_k}, \quad \text{Im } z_k \neq 0.
\]

In no event is \( e^{z_k} = 0 \). Moreover, \( e^{z_k} < 0 \) only if \( \text{Im } z_k \neq 0 \), in which case \( e^{z_k} = e^{z_k^*} \). Thus the negative eigenvalues of \( C \) must associate with elementary divisors which occur in pairs. Hence \( C \) must satisfy (*)

(ii) **Sufficiency.** Conversely, let \( C \) satisfy (*). Its eigenvalues \( \lambda_k \) are as specified by (3.6) or (3.9). For those \( \lambda_k \) that are real and negative we can write \( \lambda_k = e^{z_k} = e^{z_k^*} \), where \( z_k = \text{LOG} |\lambda_k| + i\pi \). Moreover, by the last part of (*), the corresponding elementary divisors are \((\lambda - e^{z_k})^{a_k+1} \) and \((\lambda - e^{z_k^*})^{a_k+1} \). Since, also, \( C \) is real, we may suppose that all the elementary divisors of \( C \) are given by (4.2). Consider, now, the class of matrices with elementary divisors (4.1). Clearly there exists some real matrix \( Y \) in this class. By the theorem quoted from Gantmacher, the function \( e^Y \) must be similar to \( C \), so that a real matrix \( T \) can be found such that

\[
C = T^{-1} e^Y T = \exp(T^{-1}YT).
\]

Identify \( X \) with \( T^{-1} Y T \) to confirm the sufficiency of (*).

**Theorem 2.** Let \( C \) be a real square matrix. Then the equation \( C = e^X \) has a unique real solution \( X \) if and only if (**) all the eigenvalues of \( C \) are positive real and no elementary divisor (Jordan block) of \( C \) belonging to any eigenvalue appears more than once.

**Proof.** (i) **Sufficiency.** All the solutions to \( C = e^X \) are given by (3.4), (3.5):

\[
X = SK(\text{diag}\{\log J_1, \ldots, \log J_m\})K^{-1}S^{-1},
\]

where \( \text{log } J_k \) is given by (3.14) or (3.15). Clearly, if (**) holds, \( \text{LOG } J_k \) is real, whereas \( \text{log } J_k = \text{LOG } J_k + i2\pi q_k I \) is complex and has no complex conjugate in the set \( \text{log } J_h = \text{LOG } J_h + i2\pi q_h I, \ h \neq k \). Hence, for every \( k \) the parameter \( q_k \) must be zero, and for every set of blocks (say \( J_k, J_{k+1}, \ldots, J_{k+\gamma_k} \)) which belongs to the eigenvalue \( \lambda_k \) there exists the unique set \( \text{LOG } J_k, \text{LOG } J_{k+1}, \ldots, \text{LOG } J_{k+\gamma_k} \) which belongs to the eigenvalue \( \text{LOG } \lambda_k \). Hence \( [2, \text{p. 220}] \) every \( K \) that commutes with \( J \) must also commute with \( \text{log } J \) in (3.4), and (**) is sufficient for \( X \) to be real and unique.

(ii) **Necessity.** Take the contradictions to (**) which satisfy condition (*) of Theorem 1. For example, assume \( C \) to have positive real
eigenvalues which belong to Jordan blocks that appear more than once, or assume \( C \) to have negative real eigenvalues (whose blocks must occur in pairs), or assume \( C \) to have complex conjugate eigenvalues.

Suppose, first, that \( \lambda_k \) is real and corresponds to the identical blocks \( J_k, J_{k+1} \). If in (3.14) we choose \( q_k = -q_{k+1} \) for \( \lambda_k \) positive real and \( q_k = -(1 + q_{k+1}) \) for \( \lambda_k \) negative real, we obtain the complex conjugate blocks \( \log J_k, \log J_{k+1} \). Hence a continuum set of \( K \) matrices that commuted with \( J \) will not commute with \( \log J \), and a continuum of real \( X \)'s will arise from (3.4).

Suppose now, that some pair of eigenvalues of \( C \) are complex conjugate. If they correspond to Jordan blocks that appear more than once (say \( J_k, J_{k+1} \)), then by taking \( q_k = -q_{k+1} \) in (3.15) we obtain a continuum set of \( X \)'s from (3.4). If the blocks appear only once, \( q_k \) must be zero for all \( k \) or else \( \log J_k \) will be a complex block without a conjugate. However, \( r_k \) in (3.15) can be any integer. If any two blocks (say \( J_k, J_{k+1} \)) are not identical but belong to the same eigenvalue \( \lambda_k \), the fact that \( r_k \) need not equal \( r_{k+1} \) makes it possible for \( \log J_k \) and \( \log J_{k+1} \) to belong to different eigenvalues. Hence not every \( K \) will commute with \( \log J \) and again a continuum of \( X \)'s result. Finally, if no two blocks of \( J \) belong to the same complex eigenvalue, every \( K \) that commutes with \( J \) will also commute with \( \log J \), provided the Jordan blocks for the real eigenvalues appear only once. But \( r_k \) can still be any integer, which leads to a countable infinity of \( \log J \)'s, and hence to a countable infinity of \( X \)'s. Thus (**) is necessary.

**Corollary.** Let \( C \) be a real square matrix and let \( \log C \) have more than one real solution \( X \). Then there exists an infinity of real solutions \( X \) which are

(a) Countable if all real eigenvalues of \( C \) are positive such that their Jordan blocks appear only once and \( C \) has complex eigenvalues none of which belongs to more than one Jordan block;

(b) Uncountable if any real eigenvalues of \( C \) are negative, or if any positive real eigenvalues belong to Jordan blocks that appear more than once, or if any complex conjugate eigenvalues belong to more than one Jordan block.

**References**
