A CLASS OF NATURALLY PARTLY ORDERED COMMUTATIVE ARCHIMEDEAN SEMIGROUPS WITH MAXIMAL CONDITION

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Let S be a commutative semigroup. For $a, b \in S$, we say that a *divides* b (or b is a *multiple* of a), and write $a|b$, if either $a=b$ or $ax=b$ for some $x \in S$. We say that a properly divides b if $a|b$ and b does not divide a. We call S archimedean if for all $a, b \in S$, a divides some power of b. It is known that every commutative semigroup is uniquely expressible as a semilattice of archimedean semigroups (Clifford and Preston [**2**, Theorem 4.13, p. 132], which is an easy consequence of Tamura and Kimura [**7**]).

Those commutative archimedean semigroups which are naturally totally ordered (that is, those in which the divisibility relation is a total order) have been studied by Clifford [**1**] and other authors (see Fuchs [**3**, Chapter 11] for references). Tamura [**5**], [**6**] has begun the study of those which are naturally partly ordered.

The purpose of the present paper is to determine all those commutative archimedean semigroups which satisfy the following three conditions:

(1) There is no infinite sequence of elements in which each term properly divides the one preceding it.

(2) If $a|b$ and $b|a$, then $a=b$.

(3) If $a|b$ and $a|c$, then either $b|c$ or $c|b$.

Condition (1) is essentially the maximal condition on principal ideals. It is, of course, satisfied whenever S is finite. Condition (2) states that S is naturally partly ordered. It can be shown that a commutative archimedean semigroup satisfies (2) if and only if it either contains no idempotent or contains a zero element; however, we shall not need to use this fact. The effect of (3) is to assert that every set of elements having a common divisor is naturally totally ordered. Thus (3) generalizes natural total ordering.

1. **Examples.** We begin with three examples of commutative archimedean semigroups satisfying (1), (2) and (3).

Type 1. Choose a countably infinite sequence $S_1, S_2, \cdots$ of disjoint nonempty sets. In each S_i (for $i \geqq 2$) choose an element a_i. Let $S=\bigcup S_i$. Define $xy=a_{i+j}$, if $x \in S_i$ and $y \in S_j$.

Type 2. Choose any number (finite or infinite) of finite sequences

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of nonempty sets, with all the sets disjoint. It is helpful to think of each finite sequence of sets as arranged in a column, so that we have an array of sets with any number of columns, each column consisting of a finite number of sets. For each j, let the jth column consist of $S_{1j}, S_{2j}, \cdots, S_{n_j j}$, where n_j is a positive integer depending on j. In each S_{ij} with $i \geqq 2$, choose an element a_{ij}. Let S be the union of all the S_{ij} together with two extra elements, a and 0. Define multiplication as follows. $ax = xa = 0x = x0 = 0$, for all $x \in S$. Define the products $a_{ij}x_{kl}$ and $x_{kl}a_{ij}$, where x_{kl} denotes an arbitrary element of S_{kl}, to be: $a_{i+k,j}$ if $j = l$ and $i+k \leqq n_j$, a if $j = l$ and $i+k = n_j+1$, 0 if $j = l$ and $i+k \geqq n_j+2$, and 0 if $j \neq l$. Define the products $x_{ij}x_{kl}$ and $x_{kl}x_{ij}$, where $a_{ij} \neq x_{ij} \in S_{ij}$ and $a_{kl} \neq x_{kl} \in S_{kl}$, to be: $a_{i+k,j}$ if $j = l$ and $i+k \leqq n_j$, either a or 0 (chosen arbitrarily for each pair) if either $j \neq l$ or $i+k > n_j$ (i.e., in all other cases).

We shall also include, as type 2 semigroups, the trivial cases in which $S = \{0\}$ and $S = \{a, 0\}$.

Type 3. Choose any number of disjoint semigroups, S_i, each of which is of type 2. Let S be the mutually annihilating sum (in the sense of Ljapin [4]) of these semigroups. That is, S is the union of the nonzero elements of all the semigroups, with one extra element 0 adjoined. Multiplication is defined by: $0x = x0 = 0$ for all $x \in S$, $xy = 0$ if $x \in S_i$ and $y \in S_j$ with $i \neq j$, $xy =$ the product xy in S_i if $x, y \in S_i$ and the product xy in S_i is not the zero element of S_i, $xy = 0$ if $x, y \in S_i$ and the product xy in S_i is the zero element of S_i.

2. **Preliminary results.** If S contains an element which has no multiples except itself, we shall denote this (necessarily unique) element by 0. Now let S be any commutative archimedean semigroup satisfying (2).

LEMMA 1. *If $0 \neq a \in S$ and $x \in S$, then a properly divides ax.*

PROOF. Clearly $a \mid ax$. Suppose $ax \mid a$. Then, by (2), $ax = a$. Hence, by induction on n, $ax^n = a$, so that $x^n \mid a$ for all n. Now since $a \neq 0$, we can let b be a proper multiple of a. By the archimedean property $b \mid x^n$ for some n. By transitivity of divisibility, $b \mid a$. This contradicts the fact that b is a proper multiple of a.

LEMMA 2. *If a properly divides b and $ac \neq 0$, then ac properly divides bc.*

PROOF. Write $b = ax$. Then $bc = axc = acx$. The conclusion follows by Lemma 1, with a replaced by ac.

From now on, let S be a commutative archimedean semigroup satisfying (1), (2) and (3).

LEMMA 3. *For each $a \neq 0$ in S, there exists a unique a' in S such that a properly divides a' and a' divides every proper multiple of a.*

PROOF. Let M be the set of all proper multiples of a. Since $a \neq 0$, $M \neq \varnothing$. By (1), M contains a maximal element with respect to the divisibility relation, i.e., an element a' which has no proper divisor in M. Clearly, a' is a proper multiple of a. Now let x be any proper multiple of a. Then we have $a \mid a'$ and $a \mid x$. Hence by (3) either $a' \mid x$ or $x \mid a'$. But $x \mid a'$ implies, by maximality of a', that $x = a'$. Thus in any case $a' \mid x$. The uniqueness of a' follows from (2) since any two such elements must each divide the other.

We shall call a' the *successor* of a. For convenience, we define $0' = 0$. a'' will denote the successor of a'.

LEMMA 4. *If $xa' = a'' \neq 0$, then $xa = a'$.*

PROOF. By Lemma 1, a properly divides xa. Hence $a' \mid xa$. If $a' \neq xa$, then a' would properly divide xa. Hence $a'' \mid xa$. But, by Lemma 2, xa properly divides $xa' = a''$. This is a contradiction.

We call $a \in S$ *suitable* if, whenever x properly divides a', it follows that $x' = a'$.

LEMMA 5. *For each $a \in S$, there is a suitable element which divides a'.*

PROOF. If a itself is suitable, we are finished. If not, let x_1 properly divide a', with $x_1' \neq a'$. Then x_1' properly divides a'. If x_1 is not suitable, let x_2 properly divide x_1' with $x_2' \neq x_1'$. Then x_2' properly divides x_1'. Continuing in this way, we obtain an infinite ascending sequence $a', x_1', x_2', \cdots$. This would contradict (1). Thus one of the x_i must be suitable.

LEMMA 6. *Suppose $a \in S$ is suitable, and $a'' \neq 0$. Then, for each positive integer n, $(a^n)' = a^{n+1}$.*

PROOF. Let x be such that $a'x = a''$. By Lemma 4, $ax = a'$. By Lemma 1, x properly divides a'. Since a is suitable, $x' = a'$. Thus $x'x = a'x = a''$. Now apply Lemma 4 (with a replaced by x) to obtain $x^2 = x' = a'$. Hence $a'a = x^2a = (ax)x = a'x = a''$. By Lemma 4, $a^2 = a'$. Thus, we have proved Lemma 6 for the case in which $n = 1$. We now proceed by induction on n. Assume $(a^n)' = a^{n+1}$. Clearly $(a^n)'' \mid a^{n+2}$. *Case* I: $(a^n)'' = 0$. Here, a^{n+2} is a multiple of 0, and hence is itself 0. Thus we have $a^{n+2} = (a^{n+1})'$, using the inductive assumption. *Case* II: $(a^n)'' \neq 0$. Here, write $(a^n)'' = (a^n)'y$, and apply Lemma 4 to obtain $a^ny = (a^n)' = a^{n+1}$. Then multiply by a to obtain $a^{n+1}y = a^{n+2}$. Since $a^{n+1}y = (a^{n+1})'$, we have shown that $(a^{n+1})' = a^{n+2}$.

LEMMA 7. *If a is suitable and $a|b$ and $a''\neq 0$, then $a^n=b$ for some positive integer n.*

PROOF. Let n be the least positive integer such that $b|a^n$; such integers exist by the archimedean property. If $a=b$, the lemma is trivially true. Thus, we may assume that $n\geqq 2$. Now $a|a^{n-1}$ and $a|b$. Hence, by (3), either $a^{n-1}|b$ or $b|a^{n-1}$. But the second alternative is impossible by the choice of n. Thus a^{n-1} properly divides b. By Lemma 6, $(a^{n-1})'=a^n$. Hence, by Lemma 3, $a^n|b$. Hence, by (2), $a^n=b$.

LEMMA 8. *If a is suitable, and a properly divides b, then $ab=b'$.*

PROOF. *Case* I: $a''=0$. Here, by Lemma 3, $a'|b$. Hence $0=a''|b'$. Hence $b'=0$. If $b=0$, then $ab=0$, and we are finished. If $b\neq 0$, then, by Lemma 1, b properly divides ab. Hence, by Lemma 3, $b'|ab$, so that $ab=0=b'$.

Case II: $a''\neq 0$. Here, by Lemma 7, $a^n=b$ for some n. Hence, by Lemma 6, $b'=(a^n)'=a^{n+1}=aa^n=ab$.

For each a, $b\in S$, we can consider the set M of all common multiples of a and b. By (1), M contains an element x having no proper divisor in M. By (3), x must divide every element of M, and hence is unique. We shall denote this x by $a\wedge b$.

LEMMA 9. *If a is suitable and does not divide b, and $(a\wedge b)'\neq 0$, then $ab=a\wedge b$.*

PROOF. Since ab is a common multiple of a and b, we have $a\wedge b|ab$. Suppose $a\wedge b\neq ab$. Then $a\wedge b$ would properly divide ab. Hence, by Lemma 3, $(a\wedge b)'|ab$. On the other hand, since $a|a\wedge b$, we have $ab|(a\wedge b)a=(a\wedge b)'$, using Lemma 8. Thus $ab=(a\wedge b)'$. This is impossible by Lemma 1, since b properly divides $a\wedge b$.

LEMMA 10. *If a is suitable and $(a\wedge b)'\neq 0$, then $a|b'$.*

PROOF. Suppose not. Then, by Lemma 9, $ab=a\wedge b$. Now note that $b'|a\wedge b$, so that $a\wedge b'=a\wedge b$. Hence, by Lemma 9, $ab'=ab$. By our hypothesis, $b\neq 0$. Hence b properly divides b'. Hence, by Lemma 2, $ab=0$. But, since $ab=(a\wedge b)'$, this contradicts our hypothesis.

LEMMA 11. *If $a|b$, $a|c$, $b'=c'$ and $b\neq 0\neq c$, then $b=c$.*

PROOF. By (3), one of b, c divides the other. Say $b|c$. If $b\neq c$, b would divide c properly. This would imply $c'=b'|c$. Hence $c=0$, contradicting the hypothesis.

LEMMA 12. *If a is suitable, $b'=a^n$, $c'=a^m$ and $a^{m+n-1}\neq 0$, then $bc=a^{m+n-2}$.*

PROOF. First note that $m>1$. (For if $m=1$, then $a=b'$, so that b properly divides a. Hence, since a is suitable, $b'=a'$, so that a, being its own successor, must be 0. This contradicts the last hypothesis.) Similarly $n>1$. Clearly $a''|a^3$. Thus, by the last hypothesis, we have $a''\neq 0$. Now we show that $ab=b'$. There are three cases. If $a=b$, we use Lemma 6 (with the n of Lemma 6 replaced by 1). If a properly divides b we use Lemma 8. If a does not divide b, we note that $a\wedge b=b'=a^n$. Hence, by Lemma 6, $(a\wedge b)'=a^{n+1}$. Thus, by our hypothesis, $(a\wedge b)'\neq 0$. We can now apply Lemma 9, to conclude $ab=a\wedge b=b'$. This shows that, in all cases, $ab=b'$. Similarly $ac=c'$. Next write $b'c=a^nc=a^{n-1}ac=a^{n-1}a^m=a^{m+n-1}$, and similarly $bc'=a^{m+n-1}$. By Lemma 1, b properly divides bc. Hence $b'|bc$. But a properly divides b'. Hence, a properly divides bc. Thus by Lemma 8 (replacing the b of Lemma 8 by bc) we obtain $(bc)'=abc=b'c=a^{m+n-1}$. By Lemma 6, this is $(a^{m+n-2})'$. Finally, by Lemma 11, we conclude $bc=a^{m+n-2}$.

3. Main theorem.

THEOREM. *Every commutative archimedean semigroup satisfying* (1), (2) *and* (3) *is isomorphic to a semigroup of type* 1, *type* 2 *or type* 3, *as given in* §1.

PROOF. *Case* I: S has no zero element. We shall show that S is isomorphic to a semigroup of type 1. Let s be a suitable element (which exists by Lemma 5). For all $x\in S$, x' is a multiple (by Lemma 10) and hence a power (by Lemma 7) of s. For each $n\geqq 2$, define $a_n=s^n$ and $S_{n-1}=\{x\in S: x'=s^n\}$. By Lemma 1 the powers of s are all distinct. Hence each $x\in S$ belongs to exactly one S_n. Suppose $x\in S_n$ and $y\in S_m$. Then $x'=s^{n+1}$ and $y'=s^{m+1}$. Hence, by Lemma 12, $xy=s^{m+n}=a_{m+n}$. Thus the multiplication of elements agrees with the multiplication specified for type 1 semigroups.

Case II: S has a zero element. Define an equivalence relation σ on S by: $x\,\sigma\,y$ if and only if there exists $z\neq 0$ in S such that $x|z$ and $y|z$. Let T be any σ-class together with 0. If $x, y\in T$, then xy is either 0 or else a common nonzero multiple of x and xy. Thus in any event $xy\in T$, so that T is itself a semigroup. Moreover, it is clear that S is the mutually annihilating sum (in the sense of §1) of these subsemigroups. Thus it will suffice to show that each T is of type 2. First note that T is itself a commutative archimedean semigroup satisfying (1), (2), (3), and

(4) If a and b are nonzero elements of T, then $a\wedge b\neq 0$.

Subcase A. There exists a suitable element $s\in T$ such that $s'=0$. Here, every nonzero $x\in T$ properly divides s'. Hence, by definition of suitability, $x'=s'=0$. Thus, if $x\neq s$, $x\wedge s=0$, contradicting (4).

Hence S consists either of one element 0 or of two elements 0 and a. By Lemma 1, a^2 is a proper multiple of a, so that $a^2=0$. Thus all products are 0, so that S must be one of the two trivial semigroups which were included under type 2.

Subcase B. There exists a suitable element $s\in T$ such that $s'\neq 0 = s''$. Here, define $a=s'$, $S_{11}=\{x\in T: s'\neq x\neq 0\}$. We shall show that T is a type 2 semigroup with only one set S_{11} in the array. First note that, by Lemma 1, ax is, for all $x\in T$, a proper multiple of a. But $a'=0$, so that 0 is the only proper multiple of a. Hence $ax=0$. Now suppose that $x\in S_{11}$. Then by (4) $a\wedge x\neq 0$. Hence $a\wedge x=a$, so that x properly divides a. Hence, since s is suitable, $x'=s'=a$, so that the only proper multiples of x are a and 0. This implies that the product of two elements of S_{11} is always either a or 0, as provided in the description of type 2 semigroups.

Subcase C. For all suitable $s\in T$ we have $s''\neq 0$. Let A be the set of all suitable elements of T. Define an equivalence relation σ on A by: $x\,\sigma\, y$ if and only if $x'=y'$. From each σ-class choose an element x_j. By the archimedean property, some power of x_j is 0. Let m_j be, for each j, the smallest such power. Then in the sequence $x_j, x_j^2, x_j^3, \cdots, x_j^{m_j}=0$, each term is the successor of the preceding, by Lemma 6. In particular $(x_j^{m_j-1})'=0$. By (4) $x_j^{m_j-1}$ and x_k^{m-1} have a nonzero common multiple, and hence are equal, for all j and k. Call this element a. Now let $n_j=m_j-2$, for each j. Define S_{ij}, for $i<n_j$, to be $\{x\in T: x'=x_j^{i+1}\}$. Define S_{n_jj}, for each j, by partitioning the set $\{x\in T: x'=a\}$ among the sets of the form S_{n_jj} in an arbitrary manner, subject to the provision that for all j, $x_j^{n_j}\in S_{n_jj}$. Define a_{ij}, for $2\leqq i\leqq n_j$ to be x_j^i. If $x\in T$ and $a\neq x\neq 0$, then it is clear that x belongs to at least one of the sets S_{ij}, since x' is a multiple (by Lemma 5) and hence a power (by Lemma 7) of some suitable element x, and hence is also a power of that x_j which lies in the same σ class as x; the power must be greater than 1 since x' cannot be suitable, and must be less than m_j since otherwise $x'=0$ so that $x=a$.

Now we shall show that no element of T belongs to two different sets of the form S_{ij}. For this to happen, it would be necessary to have (since we have seen that the first m_j powers of x_j are all distinct) $x_j^i=x_l^k$ with $j\neq l$, $i\leqq n_j$ and $k\leqq n_l$. Let i and k be the smallest powers of x_j and x_l which are equal to each other. Then $x_j\wedge x_l=x_j\neq a$. Thus we can apply Lemma 9 to obtain $x_jx_l=x_j^i$. Now if either i or k (say i) were greater than 2, we should have three elements, x_j, x_j^2, x_j^i, each of which would be a proper multiple of the preceding. Hence, by Lemma 2, x_jx_l properly divides $x_j^2x_l$, which in turn properly divides $x_j^ix_l$. This would contradict the fact (obtained from Lemma 8) that

$x_j^i x_l = (x_j x_l)'$. If either i or k were 1, one suitable element would divide another, and this is impossible. Thus, $i = k = 2$. But this implies that $(x_j)' = (x_l)'$, contradicting the choice of one x_j from each σ-class.

Now we must show that the multiplication in T agrees with the multiplication specified in §1 for type 2 semigroups. We have seen that 0 is the only proper multiple of a; thus, we have $ax = xa = 0$ for all $x \in T$. Now suppose $x \in S_{kl}$. Then $x_l x = x'$. (This follows from Lemmas 7 and 6 if $x_l | x$. If x_l does not divide x, then clearly $x_l \wedge x = x' = x_l^{k+1} \neq 0$, so that we can apply Lemma 9.) Hence $a_{il} = x_l^i x = x_l^{i+k}$. This shows that the definition of the products $a_{ij} x_{kl}$ and (by commutativity) $x_{kl} a_{ij}$, given for type 2 semigroups, is correct whenever $j = l$. On the other hand if $j \neq l$, we know from the preceding paragraph that $x_j \wedge x = a$. By definition of $\wedge$, $x_j x$ is either a or 0. Since x_j properly divides each a_{ij}, we conclude by Lemma 2 that $a_{ij} x = 0$. Thus, we have shown that each a_{ij} always multiplies in the right way.

Now suppose $y \in S_{ij}$ and, as before, $x \in S_{kl}$. If $j = l$ and $i + k \leqq n_j$ we can apply Lemma 12 (replacing a, b, c, m and n by x_j, x, y, $i+1$, $k+1$ respectively) to conclude $xy = yx = x_j^{i+k} = a_{i+k,j}$. If $j = l$ and $i + k > n_j$, we note that x properly divides xy by Lemma 1. Hence $x' | xy$. But $x' = x_j^{k+1}$, so that, by Lemma 7, $xy = x_j^n$ for some n. Hence $x_j^{n+1} = x_j xy = x'y = a_{k+1,l} y = 0$. But $n_j + 2$ is the smallest power of x_j which is 0. Hence $n + 1 \geqq n_j + 2$. Hence $n \geqq n_j + 1$. Since $x_j^{n_j+1} = a$, we have shown that $xy = x_j^n =$ either a or 0. Finally, if $j \neq l$, we know that $x \wedge y = a$. Hence xy is either a or 0. This completes the proof of the fact that T is isomorphic to a type 2 semigroup.

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