ON AN APPLICATION OF SLOWLY OSCILLATING FUNCTIONS

MANOJLO MARAVIĆ

1. Let $Q(x_1, \cdots, x_n)$ be a point in the n-dimensional Euclidean space and $f(Q)=f(x_1, \cdots, x_n)$ a real-valued, L-integrable function having the period 2π in each variable. Let

$$(1) \qquad f(Q) \sim \sum_{-\infty}^{+\infty} \cdots \sum_{-\infty}^{+\infty} a_{m_1 \cdots m_n} \exp\left(i \sum_{j=1}^{n} m_j x_j\right)$$

be its Fourier series where

$$(2) \quad a_{m_1 \cdots m_n} = (2\pi)^{-n} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} f(Q) \exp\left(-i \sum_{j=1}^{n} m_j x_j\right) dx_1 \cdots dx_n.$$

The spherical partial sum of order k of the multiple Fourier series (1) is defined by

$$(3) \qquad S_k(Q) = \sum_{\mu=0}^{k} A_\mu(Q),$$

where

$$(4) \qquad A(Q) = \sum_{m_1^2+\cdots+m_n^2=\mu} a_{m_1 \cdots m_n} \exp\left(i \sum_{j=1}^{n} m_j x_j\right),$$

and $A_\mu(Q)\equiv 0$ if μ cannot be represented as a sum of n squares. ϕ-mean of the spherical partial sums (3) is defined by

$$(5) \qquad S_\rho^\phi(Q) = \sum_{\nu \leq \rho} \phi\left(\frac{\nu}{\rho}\right) a_{m_1 \cdots m_n} \exp\left(i \sum_{j=1}^{n} m_j x_j\right),$$

where $\nu^2 = m_1^2 + \cdots + m_n^2$, and $\phi(t)$ is a function defined for $0 \leqq t < \infty$ for which $\phi(0)=1$. The Riesz mean $S_\rho^\delta(Q)$ of order δ of the spherical partial sums (3) is a particular case of the ϕ-mean (5) when

$$(6) \qquad \begin{aligned} \phi(t) \equiv K_\delta(t) &= (1-t^2)^\delta \quad &&\text{for } 0 \leqq t < 1, \\ &= 0 &&\text{for } t \geqq 1. \end{aligned}$$

Let $f_P(t)$ be the spherical mean of the function $f(Q)$ over a sphere whose radius is t and whose center is at the fixed point $P=(x_1^0, \cdots, x_n^0)$, i.e.

Received by the editors April 18, 1966.

$$f_P(t) = 2^{-1}\pi^{-r-1}\Gamma(r+1)\int_\sigma (\overset{0}{x}_1 + t\xi_1, \cdots, \overset{0}{x}_n + t\xi_n)\, d\sigma_\xi \tag{7}$$

where σ is the unit sphere $\xi_1^2 + \cdots + \xi_n^2 = 1$, $d\sigma_\xi$ its $(n-1)$-dimensional volume element and

$$r = (n-2)/2. \tag{8}$$

ϕ-mean $S_\rho^\phi(P)$, defined by (5), can be expressed in terms of the spherical mean $f_P(t)$ [2]:

$$S_\rho^\phi(P) = 2^{-r}[\Gamma(r+1)]^{-1}\rho\int_0^\infty f_P(t)H_\phi(t\rho)\, dt, \tag{9}$$

whenever

$$\int_0^\infty |\phi(t)|\, t^{2r+1}\, dt < \infty. \tag{10}$$

The kernel $H_\phi(w)$ is defined by

$$\begin{aligned} H_\phi(w) &= w^{-1}\int_0^\infty \phi(z/w) z^{2r+1} V_r(z)\, dz \\ &= w^{2r+1}\int_0^\infty \phi(z) z^{2r+1} V_r(zw)\, dz, \end{aligned} \tag{11}$$

where

$$V_\mu(z) = z^{-\mu}J_\mu(z), \tag{12}$$

and $J_\mu(z)$ is the Bessel function of the first kind of order μ.

Bochner proved the following lemmas [2] on the asymptotic behaviour of $H_\phi(w)$:

LEMMA A. *Let $\phi(t)$ be either the Riesz kernel $K_\delta(t)$ with $\delta > r + 1/2$ or a function in $0 \leqq t < \infty$ having the following properties*:

(i) *The inequality* (10) *holds*;

(ii) *if p is the integer defined by $-1/2 \leqq r - p < 1/2$, then $\phi(t)$ has $(p+2)$ derivatives in $0 \leqq t < \infty$, each bounded in some neighborhood of $t = 0$, such that*

$$\limsup |\phi^{(\mu)}(z) \cdot z^\gamma| < \infty, \qquad \int_0^\infty |\phi^{(\mu)}(z)\cdot z^\gamma|\, dz < \infty$$

for $\mu = 0, 1, 2, \cdots, p+2$; $0 \leqq \gamma < r + 1/2$.

Under these conditions $H_\phi(w)$ has the following properties:

$$|H_\phi(w)| \leqq A_1 w^{2r+1}, \qquad |H_\phi(w)| \leqq A_2 w^{-1-\kappa}$$

where A_1, A_2, *and* κ *are positive constants independent on* w.

LEMMA B. *If* λ *is any real number* $\geqq -1/2$ *and if a function* $\phi(t)$ *satisfies hypothesis* (ii) *of Lemma* A, *then*

$$\int_0^\infty \phi(z/w) z^{2\lambda+1} V_\lambda(z)\, dz = O(w^{-\kappa}) \quad \text{for } \kappa > 0, \text{ as } w \to \infty. \tag{13}$$

We can consider the spherical mean $f_P(t)$ as the mean of order zero, and define the spherical means of higher order $f_{P,s}(t)$ [3] by

$$\begin{aligned} f_{P,s}(t) &= 2^s\Gamma(s)[B(s, r+1)]^{-1} t^{-2r-2s} \psi_{P,s}(t) \quad \text{for } s > 0 \\ &= f_P(t) \quad \text{for } s = 0, \end{aligned} \tag{14}$$

where

$$\psi_{P,s}(t) = 2^{1-s}[\Gamma(s)]^{-1} \int_0^t (t^2 - \tau^2)^{s-1} \tau^{2r+1} f_P(\tau)\, d\tau, \qquad s > 0. \tag{15}$$

It is easy to show [3] that

$$\psi_{P,s+1}(t) = \int_0^t \tau \psi_{P,s}(\tau)\, d\tau, \tag{16}$$

$$f_{P,s}(t) = O(1) \quad \text{for } s \geqq 1, \text{ as } t \to \infty. \tag{17}$$

$S_\rho^\delta(P)$ can be expressed in terms of spherical mean of higher order and so we have [3] the generalized Bochner formula

$$S_\rho^\delta(P) = c_1 \rho^{2r+2s+2} \int_0^\infty t^{2r+2s+1} f_{P,s}(t) V_{\delta+s+r+1}(t\rho)\, dt, \tag{18}$$

for $\delta > s + r - 1/2$, where s is a positive integer, and

$$c_1 = 2^{\delta-r-s}\Gamma(1+\delta)[\Gamma(r+s+1)]^{-1}.$$

If $s = 0$, this formula holds for $\delta > r + 1/2$.

2. We shall express ϕ-mean $S_\rho^\phi(P)$ of the spherical partial sums (3) in terms of spherical mean of higher order and prove the following.

THEOREM 1. *If* s *is a positive integer and*

$$\int_0^\infty |\phi(t)|\, t^{2r+2s+1}\, dt < \infty \tag{19}$$

and $\phi(t)$ *satisfies hypothesis* (ii) *of Lemma* A, *then*

$$(20)\qquad S_\rho^\phi(P) = \frac{2^{-r-s}}{\Gamma(r+s+1)}\rho^{2r+2s+2}\int_0^\infty t^{2r+2s+1}f_{P,s}(t)F_{\phi,s}(t\rho)\,dt,$$

where

$$(21)\qquad F_{\phi,s}(t\rho) = \int_0^\infty \phi(z)z^{2r+2s+1}V_{r+s}(zt\rho)\,dz.$$

PROOF. In virtue of the properties of function $\phi(t)$ the assumption (19) implies (10). This means formula (9) holds whenever the condition (19) holds. We write (9) in the form

$$(22)\qquad S_\rho^\phi(P) = \frac{\rho^{2r+2}}{2^r\Gamma(r+1)}\int_0^\infty t^{2r+1}f_{P,0}(t)(t\rho)^{-2r-1}H_\phi(t\rho)\,dt,$$

where according to (11)

$$(t\rho)^{-2r-1}H_\phi(t\rho) = \int_0^\infty \phi(z)z^{2r+1}V_r(zt\rho)\,dz.$$

If we integrate by parts (22) s times employing (15), (16), (19), (21), and

$$(23)\qquad \frac{d}{dx}[V_\mu(x)] = -xV_{\mu+1}(x)$$

we obtain

$$(24)\quad S_\rho^\phi(P) = \frac{2^{-r}}{\Gamma(r+1)}\left[Q_s(t)\Big|_{t=0}^{t=\infty} + \rho^{2r+2s+2}\int_0^\infty t\psi_{P,s}(t)F_{\phi,s}(t\rho)\,dt\right],$$

where

$$(25)\qquad Q_s(t) = \rho^{2r+2s}\psi_{P,s}(t)\int_0^\infty \phi(z)z^{2r+2s-1}V_{r+s-1}(zt\rho)\,dz,$$

or by (14)

$$(26)\qquad Q_s(t) = \frac{B(s,r+1)}{2^s\Gamma(s)}f_{P,s}(t)\int_0^\infty \phi(z/t\rho)z^{2r+2s-1}V_{r+s-1}(z)\,dz.$$

According to (13) and (17) we get from (26)

$$(27)\qquad Q_s(t) = O[(t\rho)^{-\kappa}] = o(1),\qquad t\to\infty.$$

Further, from (25) and

(28) $|V_\mu(x)| \leqq M$ on the interval $(0,\infty)$, where M is constant,

we have

$$| Q_s(t) | \leqq M\rho^{2r+2s} | \psi_{P,s}(t) | \int_0^\infty | \phi(z) | z^{2r+2s-1}\, dz.$$

Since $\psi_{P,s}(t) \to 0$ as $t \to 0$, it follows by (19) that

$$(29) \qquad Q_s(t) = o(1), \qquad t \to 0.$$

Now by (24), (27), and (29) we obtain

$$(30) \qquad S_\rho^\phi(P) = \frac{2^{-r}}{\Gamma(r+1)} \rho^{2r+2s+2} \int_0^\infty t\psi_{P,s}(t) F_{\phi,s}(t\rho)\, dt.$$

Employing (14), (20) follows from (30) and Theorem 1 is proved.

In particular, if $\phi(t) = K_\delta(t)$, i.e. $\phi(t)$ is the Riesz kernel (5), then

$$F_{\phi,s}(t\rho) = 2^\delta \Gamma(1+\delta) V_{\delta+r+s+1}(t\rho)$$

and in this case formula (20) becomes (18).

3. A function $L(x)$ defined for $x \geqq 0$ belongs to the class of slowly oscillating functions at infinity if [4]

(a) $L(x)$ is positive and continuous in $0 \leqq x < \infty$;

(b) $\lim_{x\to\infty} [L(tx)/L(x)] = 1$ for every fixed $t > 0$.

We shall employ the following properties of slowly oscillating functions:

(a′) If $\lambda > 0$, then [4]

$$(31) \qquad x^\lambda L(x) \to \infty, \quad x^{-\lambda} L(x) \to 0, \qquad x \to \infty.$$

(b′) If $g(t)$ is such that both integrals

$$(32) \qquad \int_0^1 t^{-a} | g(t) |\, dt \quad \text{and} \quad \int_1^\infty t^a | g(t) |\, dt$$

exist for some $a > 0$, then [1]

$$(33) \qquad \int_0^\infty g(t) L(tx)\, dt \cong L(x) \int_0^\infty g(t)\, dt, \qquad x \to \infty.$$

4. Now we are going to prove a theorem which will give us the asymptotic behaviour of ϕ-mean $S_\rho^\phi(P)$ as $\rho \to \infty$, provided we know that the asymptotic behaviour of the spherical mean of higher-order $f_{P,s}(t)$ for $t \to 0$ is connected with the behaviour of a slowly oscillating function.

THEOREM 2. *Let* $\phi(t)$ *satisfy condition* (19) *and hypothesis* (ii) *of Lemma* A *and let*

$$(34) \qquad -\kappa < \alpha < 2r + 2s + 2,$$

where s is a positive integer and κ is a positive real number.

If at a fixed point $P = (x_1^0, \cdots, x_n^0)$

$$(35) \qquad f_{P,s}(t) \cong t^{-\alpha} L(1/t), \qquad t \to 0$$

where $L(x)$ *is a slowly oscillating function at infinity, then*

$$(36) \qquad S_\rho^\phi(P) \cong \beta \rho^\alpha L(\rho), \qquad \rho \to \infty$$

with

$$(37) \qquad \beta = 2^{-r-s}[\Gamma(r+s+1)]^{-1} \int_0^\infty t^{2r+2s+1-\alpha} F_{\phi,s}(t)\, dt.$$

PROOF. Since $\phi(t)$ satisfies condition (19) and hypothesis (ii) of Lemma A, then according to the Theorem 1 we can write

$$(38) \qquad \begin{aligned} S_\rho^\phi(P) &= c\rho^{2r+2s+2} \int_0^\eta t^{2r+2s+1-\alpha} L(1/t) F_{\phi,s}(t\rho)\, dt \\ &\quad + c\rho^{2r+2s+2} \int_0^\eta t^{2r+2s+1}[f_{P,s}(t) - t^{-\alpha}L(1/t)] F_{\phi,s}(t\rho)\, dt \\ &\quad + c\rho^{2r+2s+2} \int_\eta^\infty t^{2r+2s+1} f_{P,s}(t) F_{\phi,s}(t\rho)\, dt \\ &= I_1 + I_2 + I_3, \qquad c = 2^{-r-s}[\Gamma(r+s+1)]^{-1}, \end{aligned}$$

where, by assumption (35), η can be chosen so that

$$|f_{P,s}(t) - t^{-\alpha}L(1/t)| \leqq \epsilon t^{-\alpha} L(1/t), \quad \text{for } 0 \leqq t \leqq \eta.$$

Whence,

$$\begin{aligned} |I_2| &\leqq \epsilon c \rho^{2r+2s+2} \int_0^\eta t^{2r+2s+1-\alpha} |F_{\phi,s}(t\rho)| L(1/t)\, dt \\ &\leqq \epsilon c \rho^\alpha \int_{1/\eta\rho}^\infty t^{-2r-2s-3+\alpha} |F_{\phi,s}(1/t)| L(t\rho)\, dt. \end{aligned}$$

By (19) and (28) it follows from (21)

$$(39) \qquad |F_{\phi,s}(t\rho)| \leqq M_1.$$

Employing (13) we obtain from (21)

$$(40) \qquad F_{\phi,s}(t\rho) = O[(t\rho)^{-2r-2s-2-\kappa}] \quad \text{for } \kappa > 0, \text{ as } t \to \infty.$$

By virtue of (39) and (40), the function $g(t) = t^{\alpha-2r-2s-3} F_{\phi,s}(1/t)$ satisfies conditions (32) for all α defined by (34). Therefore it follows from (33)

$$\int_0^\infty t^{-2r-2s-3+\alpha} \left| F_{\phi,s}(1/t) \right| L(\rho t)\, dt = L(\rho) \int_0^\infty t^{\alpha-2r-2s-3} \left| F_{\phi,s}(1/t) \right| dt,$$

and finally

$$I_2 = o[\rho^\alpha L(\rho)], \qquad \rho \to \infty. \tag{41}$$

According to (17) and (40), we obtain

$$\left| I_3 \right| \leqq M\rho^{2r+2s+2} \int_\eta^\infty t^{2r+2s+1}(t\rho)^{-2r-2s-2-\kappa}\, dt = M^*\rho^{-\kappa},$$

i.e.

$$\left| I_3 \right| \leqq \frac{M}{\rho^{\alpha+\kappa}L(\rho)}\, \rho^\alpha L(\rho), \qquad M^* = \text{const.}$$

Since $\alpha+\kappa>0$, then by property (a′) of slowly oscillating functions we get

$$I_3 = o[\rho^\alpha L(\rho)], \qquad \rho \to \infty. \tag{42}$$

Finally, we have to estimate the integral I_1. We can write it in the form

$$I_1 = c\rho^\alpha \left(\int_0^\infty - \int_0^{1/\eta\rho} \right) t^{\alpha-2r-2s-3} F_{\phi,s}(1/t) L(t\rho)\, dt = I_{11} + I_{12}. \tag{43}$$

We have already mentioned that the function $g(t) = t^{\alpha-2r-2s-3}F_{\phi,s}(1/t)$ satisfies (32) and hence according to (33)

$$I_{11} \cong c\rho^\alpha L(\rho) \int_0^\infty t^{\alpha-2r-2s-3} F_{\phi,s}(1/t)\, dt;$$

i.e.

$$I_{11} \cong \beta\rho^\alpha L(\rho), \qquad \rho \to \infty \tag{44}$$

where β is defined by (37).

According to (40) $F(1/t) = O(t^{2r+2s+2+\kappa})$, $t \to 0$. Thus,

$$\left| I_{12} \right| \leqq M'\rho^\alpha \int_0^{1/\eta\rho} t^{\alpha+\kappa-1} L(t\rho)\, dt = M'\rho^{-\kappa} \int_0^{1/\eta} t^{\alpha+\kappa-1} L(t)\, dt.$$

Since $\alpha+\kappa>0$,

$$|I_{12}| \leqq M''\rho^{-\kappa} = \frac{M''}{\rho^{\alpha+\kappa}L(\rho)}\rho^{\alpha}L(\rho).$$

Whence, by (31) we have

$$I_{12} = o[\rho^{\alpha}L(\rho)], \qquad \rho \to \infty. \tag{45}$$

Therefore, from (43), (44), and (45) we obtain

$$I_1 \cong \beta\rho^{\alpha}L(\rho), \qquad \rho \to \infty. \tag{46}$$

The result (36) follows from (38), (41), (42), and (46). Theorem 2 is proved.

In particular, if $\phi(t)$ is the Riesz kernel $K_\delta(t)$, then [2]

$$F_{\phi,s}(t) = 2^{\delta}\Gamma(1+\delta)V_{\delta+r+s+1}(t). \tag{47}$$

Substituting (47) in (37) and using the formula

$$\int_0^{\infty} t^{\mu-1}V_\nu(t)\,dt = 2^{\mu-\nu-1}\Gamma(\mu/2)[\Gamma(1+\nu-\mu/2)]^{-1} \quad \text{for } 0<\mu<\nu+3/2,$$

we have

$$\beta = 2^{-\alpha}\Gamma(1+\delta)\Gamma[s+(n-\alpha)/2][\Gamma(s+n/2)\Gamma(1+\delta+\alpha/2)]^{-1}$$

as in Theorem 1 of [5]. Theorem 2 covers the Riesz kernel only for $\delta > r+s+1/2$. Namely, if $\phi(t)=K_\delta(t)$ and $\lambda = s+r$, then formula (13) (which was explored in proving this theorem) has the form [2]

$$w^{2r+2s+2}\int_0^{\infty}\phi(z)z^{2r+2s+1}V_{r+s}(zw)\,dz = 2^{\delta}\Gamma(1+\delta)w^{r+s+1-\delta}J_{\delta+r+s+1}(w)$$
$$= O\{w^{-[\delta-(r+s+1/2)]}\}, \qquad w \to \infty,$$

i.e. $\kappa = \delta-(r+s+1/2)$. Since $\kappa = 0$, we have $\delta > r+s+1/2$.

References

1. S. Aljančić, R. Bojanić and M. Tomić, *Sur la valeur asymptotique d'une classe des intégrals definies*, Publ. Inst. Math. Acad. Serbe Sci. **7** (1954), 81–94.

2. S. Bochner, *Summation of multiple Fourier series by spherical means*, Trans. Amer. Math. Soc. **40** (1936), 175–207.

3. K. Chandrasekharan and S. Minakshisundaram, *Typical means*, Oxford Univ. Press, Bombay, 1952.

4. J. Karamata, *Sur un mode de croissance régulière des fonctions*, Mathematica (Cluj) **4** (1930), 38–53.

5. M. Maravić, *Sur certains relations asymptotiques entre les moyennes sphériques d'ordre supérieur et les moyennes de Riesz des séries de Fourier multiples*, C. R. Acad. Sci. Paris. **258** (1964), 4407–4410.

Wayne State University