ON THE HOMOGENEITY OF INFINITE PRODUCTS OF MANIFOLDS

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Let M be the Cartesian product of a countably infinite family of compact manifolds M_i. If every M_i has a boundary, then M is homogeneous [2]. Since a manifold without boundary is homogeneous, M is also homogeneous if none of the M_i has a boundary. It follows that M is homogeneous if none or if infinitely many of the M_i have a boundary. Answering a question raised by R. D. Anderson concerning the remaining case, we prove

THEOREM. *The Cartesian product of a countably infinite family of compact manifolds of which finitely many have a boundary is not homogeneous.*

Since a finite product of manifolds with boundary is, in turn, a manifold with boundary, we may assume that the product contains precisely one manifold with boundary. The fact that the product is countable is irrelevant: our proof applies to any infinite (or finite) product. We assume our manifolds to be Hausdorff spaces. A space is *homogeneous* if for any pair of points there is a homeomorphism of the space onto itself carrying one of the points into the other. A space X is *essential* [1, p. 519] if it cannot be deformed into a proper subset, i.e., if any homotopy $h_t\colon X\to X$ with $h_0=1$ necessarily satisfies $h_1(X)=X$; here and throughout the paper, 1 stands for the identity map.

LEMMA 1. *Let X be the Cartesian product of a countable family of compact Hausdorff spaces X_i. If every finite product of spaces X_i is essential, then X itself is essential.*

PROOF. Suppose the contrary and let $h_t\colon X\to X$ satisfy $h_0=1$, $h_1(X)=C\neq X$. Then $U=X-C$ is a nonvoid open subset of X and, as such, contains a subset of the form $a_1\times\cdots\times a_n\times X_{n+1}\times X_{n+2}\times\cdots$ for a sufficiently large $n\geqq 1$ and suitable points $a_i\in X_i$, $1\leqq i\leqq n$. Select points $b_i\in X_i$ for $i\geqq n+1$, and consider the composite

$$H_t\colon\ Y\xrightarrow{j}X\xrightarrow{h_t}X\xrightarrow{p}Y$$

where $Y=\prod_{1\leqq i\leqq n}X_i$, j embeds Y as the subset $Y\times b_{n+1}\times b_{n+2}\times\cdots$

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of X, and p projects onto the first n factors. Then, $H_0=1$ and $H_1(Y)\subset p(C)$. But $a_1\times\cdots\times a_n\notin p(C)$; for, the contrary yields a point $c=a_1\times\cdots\times a_n\times c_{n+1}\times c_{n+2}\times\cdots\in C$ so that $c\in U$, and this is impossible since $C\cap U=\emptyset$. Thus, Y fails to be essential and the result is proved.

A point $x\in X$ is *unstable* [1, p. 523] if for any open neighborhood U of x in X there is a homotopy $h_t\colon X\to X$ with

$$h_0 = 1, \quad h_t\,|\,X - U = 1, \quad h_t(U) \subset U, \quad h_1(X) \neq X. \tag{1}$$

A point which is not unstable is called *stable* [1, p. 523]. We denote by $\overline{U}$ the closure of any subset U. Also, (B^n, E^n) stands for the pair consisting of the closed unit ball and its interior in Euclidean n-space.

LEMMA 2. *Let M be the Cartesian product of a countable family of compact manifolds without boundary. Let X be a compact Hausdorff space and suppose $x\in X$ has an open neighborhood U such that the pair $(\overline{U}, U)$ is homeomorphic to (B^n, E^n). Then, the point $x\times m$ is stable in $X\times M$ for any $m\in M$.*

PROOF. Let S be the quotient space obtained from X by shrinking to a point the subset $X-U$; let $f\colon X\to S$ be the identification map. Suppose $x\times m$ is unstable and let $h_t\colon X\times M\to X\times M$ be a homotopy satisfying the conditions (1) with respect to the neighborhood $U\times M$ of $x\times m$. It is readily seen that there exists a unique homotopy H_t yielding commutativity in the diagram

$$\begin{array}{ccc} S\times M & \xrightarrow{H_t} & S\times M \\ \uparrow f\times 1 & & \uparrow f\times 1 \\ X\times M & \xrightarrow{h_t} & X\times M \end{array}.$$

Obviously, $H_0=1$. There is a point $a\times b\notin h_1(X\times M)$, and it necessarily lies in $U\times M$. Hence, $f(a)\times b\notin H_1(S\times M)$ and $S\times M$ is proved to be inessential. However, S is homeomorphic to an n-sphere, any finite product of compact manifolds without boundary is a compact manifold without boundary, and any such q-dimensional manifold Q is essential since its Cech cohomology group $H^q(Q, Z_2)\neq 0$ whereas $H^q(P, Z_2)=0$ for any closed proper subset $P\subset Q$. Thus, by Lemma 1, $S\times M$ is essential and Lemma 2 is proved.

LEMMA 3. *Let X be any space and suppose Y is completely regular. If $a\in X$ is unstable, then $a\times b$ is unstable in $X\times Y$ for any $b\in Y$.*

PROOF. Let W be any neighborhood of $a\times b$ in $X\times Y$; we may assume that $W=U\times V$ where U and V are open in X and Y. Let

$h_t\colon X \to X$ be a homotopy satisfying (1) with respect to U. Select a continuous function $r\colon Y \to I$ with $r(b)=1$, $r(Y-V)=0$. Then, $a\times b$ is readily seen to be unstable using the homotopy $H_t\colon X\times Y \to X\times Y$ given by $H_t(x\times y) = h_{tr(y)}(x)\times y$.

To prove the theorem, it only remains to note that an inner point of a manifold with boundary satisfies the assumption in Lemma 2, whereas a boundary point is obviously unstable. Thus, the Cartesian product discussed has both stable and unstable points, and cannot be homogeneous.

References

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