ON A RECENT THEOREM BY H. REITER

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Let G be a locally compact group with a fixed left invariant Haar measure μ. Let us consider the following three statements concerning G. (All the linear spaces appearing below are real linear spaces.)

(M): There is a mean m on $L_\infty(G)$ such that $m({}_xf)=m(f)$ for each $f\in L_\infty(G)$ and each $x\in G$. Here a *mean* on $L_\infty(G)$ is a linear functional m on $L_\infty(G)$ such that $m(g)\geqq 0$ whenever $g\geqq 0$ and $m(1)=1$, and, for a real valued function f on G, ${}_xf$ is a function on G defined by ${}_xf(y)=f(xy)$.

(P_1): Given a positive number ϵ and a compact subset K of G, there is an element s in $\Phi=\{f:f\in L_1(G), f\geqq 0 \text{ and } \int fd\mu=1\}$ such that $\|{}_xs-s\|_1<\epsilon$ for each $x\in K$.

(J): There is a mean m on $L_\infty(G)$ such that $m(s*f)=m(f)$ for each $f\in L_\infty(G)$ and each $s\in\Phi$. Here "*" denotes the usual convolution with respect to μ (see, for instance, Hewitt-Ross [**1**]).

The property (J) was introduced recently by Hulanicki in [**2**], where it is proved that a group G satisfies (J) if and only if it satisfies (P_1). More recently Reiter proved that (M) implies (P_1) [**3**]. (The reverse implication (P_1)$\Rightarrow$(M) is simple and well known.) In this note we shall give a short proof of the implication (M)$\Rightarrow$(J), thus giving another proof to Reiter's theorem.

THEOREM. *If a locally compact group G satisfies* (M), *then it satisfies* (J).

PROOF. Let m be a mean on $L_\infty(G)$ such that $m({}_xf)=m(f)$ for each $f\in L_\infty(G)$ and each $x\in G$. Let h be a fixed member of $L_\infty(G)$ such that $h\geqq 0$, and let ϕ be a linear functional on $L_1(G)$ defined by $\phi(g)=m(g*h)$ for $g\in L_1(G)$. Then, since $|\phi(g)|=|m(g*h)|\leqq\|g*h\|_\infty\leqq\|g\|_1\cdot\|h\|_\infty$, ϕ is bounded, and $\phi({}_xg)=\phi(g)$ for each $x\in G$ because of ${}_xg*h={}_x(g*h)$. Clearly $g\geqq 0$ implies $\phi(g)\geqq 0$. Therefore, by the uniqueness of Haar integral, there is a nonnegative number $k(h)$ such that, for each g in $L_1(G)$,

$$m(g*h)=k(h)\int gd\mu. \tag{1}$$

Obviously $k(1)=1$, $k(\lambda h)=\lambda k(h)$ and $k(h+h')=k(h)+k(h')$ for $\lambda\geqq 0$ and nonnegative elements h, h' of $L_\infty(G)$. Hence k can be extended to

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be a mean (again denoted by k) on $L_\infty(G)$, and (1) is now valid for each $g \in L_1(G)$ and each $h \in L_\infty(G)$. Now take s in Φ and f in $L_\infty(G)$; then by (1) we have

$$k(f) = k(f) \int s * s d\mu = m((s * s) * f) = m(s * (s * f))$$

$$= k(s * f) \int s d\mu = k(s * f).$$

Hence k is a mean satisfying (J).

References

1. E. Hewitt and K. A. Ross, *Abstract harmonic analysis*, Vol. 1, Springer-Verlag, Berlin, 1963.

2. A. Hulanicki, *Means and Følner condition on locally compact groups*, Studia Math. (to appear).

3. H. Reiter, *On some properties of locally compact groups*, Indag. Math. **27** (1965), 697–701.

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