A DIRECT PROOF OF PORCELLI'S CONDITION FOR WEAK CONVERGENCE

R. B. DARST[1]

Effective representations of the general bounded linear functional on a Banach space B play a prominent role in functional analysis for reasons among which is a quest for efficient determinative conditions for weak convergence of a sequence of elements of B. In spaces with a variation type norm, for example $(b\, v)$, the space of functions f of bounded variation on the interval $[0, 1]$ with $f(0)=0$, and $H(S)$, the space of bounded and finitely additive set functions on an algebra S of subsets of a set X, such representations are often lacking. Nevertheless, both $(b\, v)$ and $H(S)$ can be mapped isomorphically and semi-isometrically by well known mappings onto subspaces of appropriate (M)-spaces. These embeddings yield necessary and sufficient conditions for weak convergence to zero [**1**, Théorème 5, p. 219]. Moreover, since the (L)-spaces $(b\, v)$ and $H(S)$ are sequentially weakly complete [**3**, Theorem 12], conditions for sequential weak convergence follow.

S. Leader [**4**, Theorem 16] showed that the following L_1-type conditions of Lebesgue are necessary and sufficient in order that a sequence $\{\mu_k\}$ of elements of $H(S)$ converge weakly.

(1) The sequence $\{\mu_k(E)\}$ converges for each E in S.

(2) The sequence $\{\mu_k\}$ is equi-absolutely continuous with respect to the element ϕ of $H(S)$ defined by

$$\phi(E) = \sum_{k=1}^{\infty} 2^{-k}(1 + |\mu_k|(X))^{-1}|\mu_k|(E),$$

where $|\mu_k|$ is the variation of μ_k.

P. Porcelli [**5**]–[**7**] established that the following condition

$$\text{(A)} \qquad \lim_k \left(\sum_{i=1}^{\infty} |\mu_k(E_i)| \right) = 0 \quad \text{for each sequence } \{E_i\}$$

of pairwise disjoint elements of S, is necessary and sufficient for weak convergence of the sequence $\{\mu_k\}$ to zero. He first showed the equivalent result for $(b\, v)$ by a "somewhat tedious" argument and then embedded $\{\mu_k\}$ into $(b\, v)$.

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T. H. Hildebrandt [2] then showed the following, perhaps, even easier to apply condition

$$\text{(B)} \qquad \lim_k \left(\sum_{i=1}^{\infty} \mu_k(E_i) \right) = 0 \quad \text{for each sequence } \{E_i\}$$

of pairwise disjoint elements of S, to be equivalent to condition (A). Related matters are discussed extensively in [5]–[7]

Our purpose is to give a relatively short direct proof of the following

THEOREM. *Condition* (A) *implies condition* (2).

Since a sequence of elements of $(b\ v)$ can be mapped into $H(S)$ for a suitable choice of S, this direct proof leads to the corresponding result for $(b\ v)$ in a manner which avoids becoming embroiled in the topology of $[0, 1]$. Moreover, since condition (B) follows easily from Banach's condition [1], we obtain the following result of P. Porcelli.

COROLLARY. *The sequence* $\{\mu_k\}$ *converges weakly to zero if, and only if, condition* (A) *is satisfied.*

Turning now to a proof of the theorem, we first state the following elementary consequences of condition (A).

LEMMA 1. *If condition* (A) *is satisfied,* $\{E_i\}$ *is a sequence of pairwise disjoint elements of* S, *and* $\epsilon > 0$, *then there exists a positive integer* j *such that* $\sum_{i \geq j} |\mu_k(E_i)| < \epsilon$, $k = 1, 2, \cdots$.

LEMMA 2. *If condition* (A) *is satisfied,* $\{F_i\}$ *is a decreasing sequence of elements of* S, *and* $\epsilon > 0$, *then there exists a positive integer* j *such that* $\sum_{i \geq j} |\mu_k(F_i - F_{i+1})| < \epsilon$, $k = 1, 2, \cdots$.

PROOF OF THEOREM. Let us say that (k, E) is a pair for (δ, ϵ), $\delta > 0$, $\epsilon > 0$, if $\phi(E) < \delta$ and $|\mu_k(E)| \geq \epsilon$. Let $\delta(k, \epsilon)$ be a positive number such that $\phi(E) < \delta(k, \epsilon)$ implies that $|\mu_j(E)| < \epsilon$, $j = 1, 2, \cdots, k$. Suppose condition (2) is not satisfied. Then there is a positive number ϵ such that for each positive number δ there is a pair (k, E) for $(\delta, 2\epsilon)$. Let (k_1, E_1) be a pair for, say, $(1, 2\epsilon)$. Let (k_2, E_2) be a pair for $(\delta(k_1, \epsilon \cdot 2^{-2}), 2\epsilon)$ and, proceeding inductively, let (k_{i+1}, E_{i+1}) be a pair for $(\delta(k_i, \epsilon \cdot 2^{-(i+1)}), 2\epsilon)$. At this point, let's relabel the sequence $\{\mu_{k_i}\}$ as $\{\mu_i\}$ and record what we have obtained thus far:

(i) $|\mu_i(E_i)| \geq 2\epsilon$ and

(ii) if $E \subset E_i$, then $|\mu_j(E)| < \epsilon/2^i$, $j = 1, 2, \cdots, i-1$.

Let $F_1 = E_1$. If there exists an integer i greater than one such that $|\mu_i(F_1 \cap E_i)| > \epsilon/2$, let i_1 be the least such integer and let $F_2 = F_1 - E_{i_1}$. Then if there exists an integer i greater than i_1 such that $|\mu_i(F_2 \cap E_i)|$

$>\epsilon/2$, let i_2 be the least such integer and let $F_3=F_2-E_{i_2}$. If this process were not to stop, we would obtain, in contradiction to Lemma 2, a decreasing sequence $\{F_p\}$ of elements of S such that $|\mu_{i_p}(F_p-F_{p+1})|=|\mu_{i_p}(F_p\cap E_{i_p})|>\epsilon/2$. Hence, there exist least positive integers j_1 and p_1 such that if $i>p_1$ then $|\mu_i(F_{j_1}\cap E_i)|\leqq\epsilon/2$. In order to simplify what follows, we denote by H_1 the set F_{j_1} of the preceding sentence, let $\mu_i'=\mu_{p_1+i}$ and let $E_i'=E_{p_1+i}-H_1$. Then

(iii) $|\mu_1(H_1)|\geqq 2\epsilon-\epsilon/2$,

(iv) $|\mu_i'(E_i')|\geqq 2\epsilon-\epsilon/2$, and

(v) $|\mu_j'(E)|<\epsilon/2^{p_1+i}\leqq\epsilon/2^{(i+1)}$ if $E\subset E_i'$ and $j<i$.

Let $F_1'=E_1'$. Proceeding as before, there exist least positive integers j_2 and p_2 such that if $i>p_2$, then $|\mu_i'(F_{j_2}'\cap E_i')|<\epsilon/2^2$ and $|\mu_1'(F_{j_2}')|\geqq 2\epsilon-\epsilon/2-\epsilon/2^2$. Denoting by H_2 the set F_{j_2}', letting $E_i^2=E_{p_2+i}'-H_2$ and $\mu_i^2=\mu_{p_2+i}'$ we obtain

(vi) $H_1\cap H_2=\emptyset$,

(vii) $|\mu_{p_1+1}(H_2)|\geqq 2\epsilon-\epsilon/2-\epsilon/2^2$,

(viii) $|\mu_i^2(E_i^2)|\geqq 2\epsilon-\epsilon/2-\epsilon/2^2$, and

(ix) $|\mu_j^2(E)|<\epsilon/2^{p_1+p_2+i}<\epsilon/2^{i+2}$ if $E\subset E_i^2$ and $j<i$.

Thus, leading next to a set H_3 such $H_3\cap(H_1\cup H_2)=\emptyset$, and $|\mu_{p_1+p_2+1}(H_3)|\geqq 2\epsilon-\epsilon/2-\epsilon/2^2-\epsilon/2^3$ and eventually, letting $q_i=p_1+p_2+\cdots+p_i+1$, to a sequence $\{H_i\}$ of pairwise disjoint elements of S such that $|\mu_{q_i}(H_{i+1})|>\epsilon$ which implies that if condition (2) is not satisfied, then condition (A) is not satisfied.

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Purdue University