Effective representations of the general bounded linear functional on a Banach space $B$ play a prominent role in functional analysis for reasons among which is a quest for efficient determinative conditions for weak convergence of a sequence of elements of $B$. In spaces with a variation type norm, for example $(b\ v)$, the space of functions $f$ of bounded variation on the interval $[0, 1]$ with $f(0) = 0$, and $H(S)$, the space of bounded and finitely additive set functions on an algebra $S$ of subsets of a set $X$, such representations are often lacking. Nevertheless, both $(b\ v)$ and $H(S)$ can be mapped isomorphically and semi-isometrically by well known mappings onto subspaces of appropriate $(M)$-spaces. These embeddings yield necessary and sufficient conditions for weak convergence to zero [1, Théorème 5, p. 219]. Moreover, since the $(L)$-spaces $(b\ v)$ and $H(S)$ are sequentially weakly complete [3, Theorem 12], conditions for sequential weak convergence follow.

S. Leader [4, Theorem 16] showed that the following $L_1$-type conditions of Lebesgue are necessary and sufficient in order that a sequence $\{\mu_k\}$ of elements of $H(S)$ converge weakly.

1. The sequence $\{\mu_k(E)\}$ converges for each $E$ in $S$.
2. The sequence $\{\mu_k\}$ is equi-absolutely continuous with respect to the element $\phi$ of $H(S)$ defined by

$$\phi(E) = \sum_{k=1}^{\infty} 2^{-k}(1 + |\mu_k| (X))^{-1} |\mu_k| (E),$$

where $|\mu_k|$ is the variation of $\mu_k$.

P. Porcelli [5]–[7] established that the following condition

(A) $\lim_k \left( \sum_{i=1}^{\infty} |\mu_k(E_i)| \right) = 0$ for each sequence $\{E_i\}$

defined of pairwise disjoint elements of $S$, is necessary and sufficient for weak convergence of the sequence $\{\mu_k\}$ to zero. He first showed the equivalent result for $(b\ v)$ by a "somewhat tedious" argument and then embedded $\{\mu_k\}$ into $(b\ v)$.
T. H. Hildebrandt [2] then showed the following, perhaps, even easier to apply condition

(B) \( \lim_{k} \left( \sum_{i=1}^{\infty} \mu_k(E_i) \right) = 0 \) for each sequence \( \{E_i\} \)

of pairwise disjoint elements of \( S \), to be equivalent to condition (A). Related matters are discussed extensively in [5]–[7].

Our purpose is to give a relatively short direct proof of the following

**Theorem.** Condition (A) implies condition (2).

Since a sequence of elements of \( (b^\dagger,v) \) can be mapped into \( H(S) \) for a suitable choice of \( S \), this direct proof leads to the corresponding result for \( (b^\dagger,v) \) in a manner which avoids becoming embroiled in the topology of \([0,1]\). Moreover, since condition (B) follows easily from Banach's condition [1], we obtain the following result of P. Porcelli.

**Corollary.** The sequence \( \{\mu_k\} \) converges weakly to zero if, and only if, condition (A) is satisfied.

Turning now to a proof of the theorem, we first state the following elementary consequences of condition (A).

**Lemma 1.** If condition (A) is satisfied, \( \{E_i\} \) is a sequence of pairwise disjoint elements of \( S \), and \( \epsilon > 0 \), then there exists a positive integer \( j \) such that \( \sum_{i \leq j} |\mu_k(E_i)| < \epsilon, \ k = 1, 2, \ldots. \)

**Lemma 2.** If condition (A) is satisfied, \( \{F_i\} \) is a decreasing sequence of elements of \( S \), and \( \epsilon > 0 \), then there exists a positive integer \( j \) such that \( \sum_{i \leq j} |\mu_k(F_i - F_{i+1})| < \epsilon, \ k = 1, 2, \ldots. \)

**Proof of Theorem.** Let us say that \( (k, E) \) is a pair for \( (\delta, \epsilon) \), \( \delta > 0, \epsilon > 0 \), if \( \phi(E) < \delta \) and \( |\mu_k(E)| \geq \epsilon \). Let \( \delta(k, \epsilon) \) be a positive number such that \( \phi(E) < \delta(k, \epsilon) \) implies that \( |\mu_j(E)| < \epsilon, \ j = 1, 2, \ldots, k \). Suppose condition (2) is not satisfied. Then there is a positive number \( \epsilon \) such that for each positive number \( \delta \) there is a pair \( (k, E) \) for \( (\delta, 2\epsilon) \). Let \( (k_1, E_1) \) be a pair for, say, \( (1, 2\epsilon) \). Let \( (k_2, E_2) \) be a pair for \( (\delta(k_1, \epsilon \cdot 2^{-2}), 2\epsilon) \) and, proceeding inductively, let \( (k_{i+1}, E_{i+1}) \) be a pair for \( (\delta(k_{i+1}, \epsilon \cdot 2^{-i+1}), 2\epsilon) \). At this point, let's relabel the sequence \( \{\mu_{k_i}\} \) as \( \{\mu_i\} \) and record what we have obtained thus far:

(i) \( |\mu_i(E_i)| \geq 2\epsilon \)

(ii) if \( E \subseteq E_i \), then \( |\mu_j(E)| < \epsilon / 2^i, \ j = 1, 2, \ldots, i - 1 \).

Let \( F_1 = E_1 \). If there exists an integer \( i \) greater than one such that \( |\mu_i(F_1 \cap E_i)| > \epsilon / 2 \), let \( i_1 \) be the least such integer and let \( F_2 = F_1 - E_{i_1} \). Then if there exists an integer \( i \) greater than \( i_1 \) such that \( |\mu_i(F_2 \cap E_i)| \geq 2\epsilon \)
$>\epsilon/2$, let $i_2$ be the least such integer and let $F_3 = F_2 - E_{i_2}$. If this process were not to stop, we would obtain, in contradiction to Lemma 2, a decreasing sequence $\{F_p\}$ of elements of $S$ such that $|\mu_i(F_p - F_{p+1})| > \epsilon/2$. Hence, there exist least positive integers $j_1$ and $p_1$ such that if $i > p_1$ then $|\mu_i(F_j \cap E_i)| \leq \epsilon/2$. In order to simplify what follows, we denote by $H_1$ the set $F_{j_1}$ of the preceding sentence, let $\mu^i = \mu_{p_i+i}$ and let $E^i_i = E_{p_i+i} - H_1$. Then

(iii) $|\mu^i_1(H_1)| \geq 2\epsilon - \epsilon/2$,

(iv) $|\mu^i(E^i_i)| \geq 2\epsilon - \epsilon/2$, and

(v) $|\mu^i(E)| < \epsilon/2^{p_1+i} \leq \epsilon/2^{(i+1)}$ if $E \subseteq E^i_i$ and $j < i$.

Let $F_i = E^i_i$. Proceeding as before, there exist least positive integers $j_2$ and $p_2$ such that if $i > p_2$, then $|\mu^i_2(F_j \cap E^i_i)| < \epsilon/2^2$ and $|\mu^i(F^i_i)| \geq 2\epsilon - \epsilon/2 - \epsilon/2^2$. Denoting by $H_2$ the set $F^i_i$, letting $E^i_i = E_{p_i+i} - H_2$ and $\mu^2_i = \mu_{p_2+i}$ we obtain

(vi) $H_1 \cap H_2 = \emptyset$,

(vii) $|\mu_{p_1+i}(H_2)| \geq 2\epsilon - \epsilon/2 - \epsilon/2^2$,

(viii) $|\mu^2_i(E^2_i)| \geq 2\epsilon - \epsilon/2 - \epsilon/2^2$, and

(ix) $|\mu^2(E)| < \epsilon/2^{p_1+p_2+i} < \epsilon/2^{i+2}$ if $E \subseteq E^2_i$ and $j < i$.

Thus, leading next to a set $H_3$ such $H_3 \cap (H_1 \cup H_2) = \emptyset$, and $|\mu_{p_1+p_2+i}(H_3)| \geq 2\epsilon - \epsilon/2 - \epsilon/2^2 - \epsilon/2^3$ and eventually, letting $q_i = p_1 + p_2 + \cdots + p_i + 1$, to a sequence $\{H_i\}$ of pairwise disjoint elements of $S$ such that $|\mu_{q_i}(H_{i+1})| > \epsilon$ which implies that if condition (2) is not satisfied, then condition (A) is not satisfied.

**Bibliography**


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