

A THEOREM ON RECURSIVELY ENUMERABLE CLASSES AND SPLINTERS¹

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Terminology. Except in a few cases which will be clear from their context, we reserve the word "class" to mean a set whose members are subsets of the set of nonnegative integers. We reserve the word "set" to mean a subset of N , the set of nonnegative integers. A class \mathcal{C} of recursively enumerable (r.e.) sets is r.e. just in case there is an effective enumeration of indices for all members of \mathcal{C} . A class \mathcal{g} of r.e. sets is called *immune* if it has no infinite r.e. subclass.

1. Introduction. The chief purpose of this paper is to give some examples which show how the theory of r.e. classes differs from the theory of r.e. sets.

In [2], Friedberg has shown that the class of all recursively enumerable sets is r.e. without repetition. In [3], Pour-El and Howard have provided a structural criterion which enables them to show that various r.e. classes are r.e. without repetition. It is clear that any such class must have proper infinite subclasses which are again r.e. without repetition. In [4], Pour-El and Putnam have given examples of r.e. classes which are not r.e. without repetition. They have pointed out that to obtain an infinite r.e. class having the property that every element is repeated infinitely often in every recursive enumeration of an infinite subclass of the class it would suffice to construct an infinite r.e. class which has no proper infinite r.e. subclass. *In this paper we construct an infinite r.e. class, \mathcal{C} , of r.e. sets which has no proper infinite r.e. subclass.* Our proof uses the existence of maximal sets, [2], together with the technique which Friedberg has used to split nonrecursive r.e. sets into disjoint nonrecursive r.e. subsets, [2].

In [1, pp. 53–54], Dekker, Myhill, and Shoenfield construct two r.e. classes whose intersection is not r.e. Using techniques in [4], it is easy to obtain two r.e. classes whose intersection is immune. Here (with the aid of the referee) we show that for the r.e. class \mathcal{C} mentioned above, there exists an $\alpha^* \notin \mathcal{C}$ such that for every $\alpha \in \mathcal{C}$, $\mathcal{C}_\alpha = (\mathcal{C} - \{\alpha\}) \cup \{\alpha^*\}$ is r.e., but $\mathcal{C} \cap \mathcal{C}_\alpha (= \mathcal{C} - \{\alpha\})$ is immune.

In [4], Pour-El and Putnam use their examples of r.e. classes to

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obtain applications to r.e. sequences of formal theories. Our result together with their techniques yields similar applications: e.g., there exists a r.e. sequence \mathcal{S} of theories of identity (theories of Abelian groups, field theories, ring theories, integral domain theories, etc.) such that if $T \in \mathcal{S}$, then there is a theory T^* of identity (Abelian groups, etc.) such that $T^* \notin \mathcal{S}$, $\mathcal{S}_T = (\mathcal{S} - \{T\}) \cup \{T^*\}$ is r.e., but $\mathcal{S} \cap \mathcal{S}_T$ is immune. (In fact, just as with α^* and α in the preceding paragraph, T^* may be chosen independently of T .)

We find it convenient to use splinters to visualize our construction of \mathcal{R} . Doing this enables us to incidentally prove the existence of a one-one total recursive function which has no recursive splinters. I.e., there is a one-one Turing machine which is defined for all inputs but which has the property that, no matter what initial input is taken, if the machine is successively given its outputs as new inputs, there is no decision procedure for determining whether or not an integer belongs to the set of outputs.

2. Definitions. A *component* of a partial recursive function f is a set of the form

$$\{y \mid \exists i, j \geq 0 \ni f^i(y) = f^j(x_0)\}$$

for some integer x_0 . We will use the notation

$$\langle x, f \rangle \text{ for } \{f^i(x) \mid i \in N \text{ and } f^i(x) \text{ is defined}\},$$

where N is the set of all nonnegative integers. If f is a total recursive function, $\langle x, f \rangle$ is called an f -splinter, (Ullian). If f is one-one and $f^{-1}(x)$ is not defined, we shall call x the *head* of $\langle x, f \rangle$, and we shall say that $\langle x, f \rangle$ is headed.

3. Results. Our first lemma provides the motivation for attempting the construction given in the theorem which follows. The results mentioned in the introduction are then trivial consequences of the lemma and the theorem.

LEMMA. *Suppose M is a maximal set and f is a one-one total recursive function whose range is M . Suppose further that every component of f intersects M' and that no splinter of f is recursive. Then the class, \mathcal{R} , of all headed f -splinters is an infinite r.e. class which has no proper infinite r.e. subclass.*

PROOF. Since each component intersects M' and M is the range of f , the components of f are exactly the headed f -splinters, and each of these has a head in M' . Each headed f -splinter intersects M' exactly once. Since, given x , we can effectively enumerate the component of

f of which x is a member, \mathcal{K} is an infinite r.e. class of r.e. sets. From the maximality of M , it follows that any infinite r.e. subclass of \mathcal{K} would have to contain (indices for) all but finitely many members of \mathcal{K} . But the sum set of these finitely many members of \mathcal{K} is certainly r.e., and if its complement were r.e., this sum set would be recursive. This in turn would imply that each of these finitely many members of \mathcal{K} would be recursive. This is a contradiction unless this finite subclass of \mathcal{K} is empty, i.e. unless every infinite r.e. subclass of \mathcal{K} is in fact \mathcal{K} itself.

THEOREM. *Let R_u be any nonrecursive r.e. set. Then there exists a one-one total recursive function f whose range is R_u and which has the property that every component of f is a nonrecursive splinter with a head in R'_u .*

REMARK. Clearly none of the splinters of f can be recursive, for they all are contained in components of f and hence differ from the headed f -splinters by finite sets.

PROOF OF THEOREM. Following Friedberg's notation in Theorem 1 of [2], we shall denote the e th r.e. set by R_e . We define the function f by proceeding in steps, letting B^a denote the set of integers placed in the set B no later than Step a . We say that the number e satisfies Condition i at Step a if $R_e^a \cap \langle i, f \rangle^a$ is not empty *or* if we have specified at an earlier step that e satisfies Condition i (because i is found to be in R_u). We assume without loss of generality that $0 \in R'_u$. For all sets B in the construction, $B^0 = \emptyset$.

Step a ($a \geq 1$). If there is no member of $R'_u - R_u^{a-1}$, go on to Step $a+1$. Otherwise let $\{n\} = R'_u - R_u^{a-1}$.

If there is no e with $n \in R_e^a$ while e fails to satisfy Condition i for some $i \leq \min(e, n-1)$, we let

$$j = (\mu y)[f^y(0) \text{ is not yet defined}],$$

and we define $f^j(0) = n$. This makes all e satisfy Condition n , and we go on to Step $a+1$.

If there is an e with $n \in R_e^a$ while e fails to satisfy Condition i for some $i \leq \min(e, n-1)$, we let e_0 be the smallest such e and i_0 be the smallest such i (for e_0). We *attack* e_0 by defining $f^j(i_0) = n$, where

$$j = (\mu y)[f^y(i_0) \text{ is not yet defined}].$$

This makes e_0 satisfy Condition i_0 , and it makes all e satisfy Condition n . We go on to Step $a+1$.

This completes the description of the construction of f . Clearly each number e gets attacked at most $e+1$ times.

Now an immediate predecessor under f of an integer n is defined when and only when n appears in $R_u^a - R_u^{a-1}$. Thus f is a one-one partial recursive function whose range is R_u .

We now wish to show that for every integer n , $f^{-k}(n)$ is defined for at most finitely many integers k . Suppose that at the beginning of Step $a+1$, $n \in \langle m, f \rangle^a$ where $f^{-1}(m)$ has not yet been defined. Then in order to define $f^{-1}(m)$ during Step $a+1$, we must find m in $R_u^{a+1} - R_u^a$ and define $f^j(i_0) = m$ for some j and for some $i_0 < m$. This puts n in $\langle i_0, f \rangle^{a+1}$ where $f^{-1}(i_0)$ is not yet defined. Since $i_0 < m$, this can happen only finitely often, so $f^{-k}(n)$ is defined for only finitely many values of k . Since $f^{-1}(n)$ is defined for all $n \in R_u$, it follows that every component of f is a set of the form $\langle i, f \rangle$ for some $i \in R_u'$. If f is a total recursive function, this means that every component of f is a splinter with a head in R_u' .

It remains to prove that f is a total recursive function and that none of the sets $\langle i, f \rangle$ for $i \in R_u'$ is recursive. It suffices to prove only the latter, for if $f(y)$ were not defined, recalling that $y \in \langle i, f \rangle$ for some $i \in R_u'$, we would have that $\langle i, f \rangle$ would be finite and hence recursive. We may thus complete the proof by showing that for each $i \in R_u'$, there is no r.e. set R_c such that $\langle i, f \rangle \subseteq R_c' \subseteq R_u \cup \{i\}$. To prove Proposition 2, which follows, it will be helpful to establish a slightly stronger relation. For each finite set F and for each $i \in R_u'$, there is no r.e. set R_c such that

$$(I) \quad \langle i, f \rangle \subseteq R_c' \cup F \subseteq R_u \cup \{i\} \cup F.$$

Suppose on the contrary that such an i , F , and R_c exist. $R_c' \cup F$ equals $(R_c - F)'$; let $R_d = R_c - F$. Since R_d has infinitely many indices, we may assume that $d > i$. Clearly d never satisfies Condition i . Let a_0 be a step of the construction after which no $e \leq d$ ever gets attacked. Now if any $n > i$ belongs to R_d^a and is listed in $R_u^a - R_u^{a-1}$ for $a \geq a_0$, then some $e \leq d$ must be attacked, and this cannot occur.

Let u_0, \dots, u_k be those members of R_u' which are either less than or equal to i or which belong to F . Let u_{k+1}, \dots, u_m be the members of R_u' which are listed in R_d before Step a_0 . Recall that $R_u' - (\{i\} \cup F) \subseteq R_d$. Then

$$y \in R_u' \text{ if and only if } y \in \{u_0, \dots, u_m\} \text{ or } y > i \text{ and } y \notin R_u^a \text{ and } y \text{ is listed in } R_d^a - R_d^{a-1} \quad (a \geq a_0).$$

This shows that R_u' would be r.e., and hence that R_u would be recursive, a contradiction.

REMARK (ADDED MAY 1, 1966). Let R_u be any nonrecursive r.e.

set, let f be constructed as in the preceding proof, and let \mathcal{H} be the class of all headed f -splinters. Then relation (I) implies that any proper r.e. subclass of H differs infinitely from \mathcal{H} .

COROLLARY 1. *There is an infinite r.e. class of r.e. sets which has no proper infinite r.e. subclass.*

COROLLARY 2. *There is a one-one total recursive function f which has no recursive splinters and for which all "splinters" of the partial recursive function f^{-1} are finite. (I.e., all the sets $\langle i, f^{-1} \rangle$ for $i \in N$ are finite.)*

COROLLARY 3. *There is an infinite r.e. class \mathcal{H} of r.e. sets such that if $\alpha \in \mathcal{H}$, then there exists an $\alpha^* \notin \mathcal{H}$ with $\mathcal{H}_\alpha = (\mathcal{H} - \{\alpha\}) \cup \{\alpha^*\}$ r.e., but $\mathcal{H} \cap \mathcal{H}_\alpha (= \mathcal{H} - \{\alpha\})$ is immune.*

PROOF. Let R_u be a maximal set and let f and \mathcal{H} be as in the lemma and theorem. Let $\alpha \in \mathcal{H}$ and let a_0 be the head of α . Take $\alpha^* = \langle f(a_0), f \rangle$. Since, given x , we can effectively enumerate all members of the component of x but omitting a_0 , \mathcal{H}_α is r.e. The result now follows from the lemma and the theorem. (The referee has pointed out that, employing a technique of [4], we may instead take α^* to be N and thus obtain the stronger result: For all $\alpha \in \mathcal{H}$, $\mathcal{H}_\alpha = (\mathcal{H} - \{\alpha\}) \cup \{N\}$ is r.e., but $\mathcal{H} \cap \mathcal{H}_\alpha (= \mathcal{H} - \{\alpha\})$ is immune. The same trick yields an improved version of Proposition 2.)

If \mathcal{S} and \mathcal{H} are r.e. classes such that $\mathcal{S} \subseteq \mathcal{H}$ and $\mathcal{H} - \mathcal{S}$ is infinite, then for any finite class $\mathcal{F} \subseteq \mathcal{H} - \mathcal{S}$, $\mathcal{S} \cup \mathcal{F}$ is r.e. Thus, to construct r.e. classes with only finitely many proper infinite r.e. subclasses, all of the infinite r.e. subclasses must differ from the given classes by only finitely many members.

PROPOSITION 1. *Let n be any integer which can be written as a product of integers of the form 2^m or $2^m - 1$. Then there exists a r.e. class of r.e. sets which has exactly n distinct infinite r.e. subclasses.*

PROOF. Let R_u be a maximal set and let f and \mathcal{H} be as in the lemma and theorem. Let r_0, r_1, \dots, r_k be any integers with $r_0 \geq 0$ and $r_j \geq 1$ for $1 \leq j \leq k$. Let $\mathcal{H} = \{\alpha_0, \alpha_1, \dots, \alpha_k, \alpha_{k+1}, \dots\}$, with $\alpha_j = \langle a_j, f \rangle$ for all j . Define

$$\mathcal{H}^* = \mathcal{H} \cup \{ \{ f^i(a_0) \} \mid 1 \leq i \leq r_0 \} \cup \{ \langle f^i(a_j), f \rangle \mid 1 \leq j \leq k; 1 \leq i \leq r_j - 1 \}.$$

\mathcal{H}^* is clearly r.e., and one can prove that \mathcal{H}^* has $2^{r_0} \cdot (2^{r_1} - 1) \cdot \dots \cdot (2^{r_k} - 1)$ infinite r.e. subclasses. The crucial facts, which are verified as in the first part of the proof of the theorem, are these: every infinite r.e. subclass of \mathcal{H}^* includes $\langle a_0, f \rangle$ and may include any com-

bination at all of $\{f(a_0)\}, \dots, \{f^{r_0}(a_0)\}$, and for each $1 \leq j \leq k$ every infinite r.e. subclass of \mathcal{R}^* includes at least one of $\langle a_j, f \rangle, \dots, \langle f^{r_j-1}(a_j), f \rangle$ but may contain any combination of these in which at least one is included.

We obtain from this proposition the existence of r.e. classes with exactly 1, 2, 3, 4, 6, 7, 8, 9, 12, \dots , infinite r.e. subclasses. *Ad hoc* arguments provide other values; e.g. if R_u is maximal and if $\mathcal{R} = \{\alpha_0, \alpha_1, \alpha_2, \dots\}$ is as in the theorem and lemma, then $\mathcal{R}^* = \mathcal{R} \cup \{\alpha_0 \cup \alpha_1\}$ has exactly five infinite r.e. subclasses; it is easily seen how to combine this technique with the techniques used in the proof of the preceding proposition to obtain r.e. classes with 10, 15, 20, 25, \dots infinite r.e. subclasses, (but not, e.g., 55). We would like to know whether all positive integers are possible values for the number of infinite r.e. subclasses of r.e. classes. (Added May 1, 1966: This question has been answered affirmatively by John Florence, a student of A. H. Lachlan.)

PROPOSITION 2. *There exists a r.e. sequence \mathcal{S} of theories of identity (theories of Abelian groups, field theories, ring theories, integral domain theories, etc.) such that if $T_k \in \mathcal{S}$, then there exists a theory T_k^* of identity (of Abelian groups, etc.) such that $T_k^* \notin \mathcal{S}$ and $\mathcal{S}_k = (\mathcal{S} - \{T_k\}) \cup \{T_k^*\}$ is r.e., but $\mathcal{S} \cap \mathcal{S}_k (= \mathcal{S} - \{T_k\})$ is immune.*

PROOF. Let R_c be the set of all prime numbers and let R_u be a maximal set such that $R'_c \subseteq R_u$. Let \mathcal{R} and f be as in the theorem and lemma, and let $\mathcal{R} = \{\alpha_0, \alpha_1, \dots, \alpha_n, \dots\}$. Following [4], let Φ_n express the proposition that there are exactly n elements in the domain. For theories of identity, the axioms of the theory T_k are exactly those sentences $\neg \Phi_i$ for which $i \in \alpha_k$. From the relation (I) in the proof of the theorem, we see that each α_k contains infinitely many primes. Let p_k be some prime number belonging to α_k . We define α_k^* to be $\alpha_k - \{p_k\}$, and we then take as axioms for the theory T_k^* exactly those sentences $\neg \Phi_i$ for which $i \in \alpha_k^*$.

For the various algebraic theories, we take as axioms for T_k and T_k^* not only the axioms of identity as given above but also the appropriate algebraic axioms. In each case $\mathcal{S} = \{T_k \mid k \geq 0\}$, and for each $T_k \in \mathcal{S}$ we take $\mathcal{S}_k = (\mathcal{S} - \{T_k\}) \cup \{T_k^*\}$. The techniques which enable us to now verify the proposition are straightforward, and the reader may find them applied to obtain similar results in [4].

As a final remark, note that Theorem 13 of [4] is applicable to each of the sequences \mathcal{S} mentioned in the preceding proposition. Thus each of these sequences has an interesting superclass which is r.e. without repetitions.

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