A REPRESENTATION OF THE WREATH PRODUCT OF TWO TORSION-FREE ABELIAN GROUPS IN A POWER SERIES RING

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1. Introduction.

1.1. Let $\boldsymbol{Q}$ be the field of rational numbers, let Λ be a well-ordered index set and let $\boldsymbol{R}$ be the power series ring over $\boldsymbol{Q}$ in the generators $x_{i\lambda}$ ($i=1, 2, \lambda\in\Lambda$), subject to the defining relations

$$(1) \qquad x_{2\lambda}x_{1\mu} = x_{1\lambda}x_{1\mu}x_{2\nu} = x_{i\lambda}x_{i\mu} - x_{i\mu}x_{i\lambda} = 0 \quad (i = 1, 2; \lambda, \mu, \nu \in \Lambda).$$

It is easy to see that every element s ($s\neq 0$) in the multiplicative semigroup S of $\boldsymbol{R}$ generated by the $x_{i\lambda}$ can be represented uniquely in the form

$$(2) \qquad s = x_{i_1,\lambda_1}x_{i_2,\lambda_2}\cdots x_{i_r,\lambda_r} \qquad (1 \leqq i_j \leqq 2, \lambda_j \in \Lambda),$$

where

(i) if $i_1=1$ and $i_2=1$, then $i_j=1$ for $j=1, 2, \cdots, r$, and $\lambda_1\leqq\lambda_2 \leqq \cdots \leqq\lambda_r$ (where $\leqq$ is of course the order relation in Λ),

(ii) if $i_1=1$ and $i_2=2$, then $i_j=2$ for $j\geqq 2$ and $\lambda_2\leqq \cdots \leqq\lambda_r$,

(iii) if $i_1=2$ then $i_j=2$ for $j\geqq 1$ and $\lambda_1\leqq\lambda_2\leqq \cdots \leqq\lambda_r$. We term as ($a\in\boldsymbol{Q}$, $a\neq 0$) a monomial of degree r, where s is given by (2).

Thus every element ρ in $\boldsymbol{R}$ is an infinite sum

$$\rho = {}_0\rho + {}_1\rho + {}_2\rho + \cdots$$

where ${}_0\rho\in\boldsymbol{Q}$ and ${}_i\rho$, the so-called *homogeneous component of* ρ *of degree* i, is either 0 or a finite sum of monomials of degree i. The *order* of ρ is the first i for which ${}_i\rho\neq 0$, with infinity the order of 0.

It is easy to see that if R_i is the set of elements of $\boldsymbol{R}$ of order at least i, then R_i is an ideal of $\boldsymbol{R}$ and

$$(3) \qquad \bigcap_{i=1}^{\infty} R_i = 0.$$

Let A be the set of those elements α in $\boldsymbol{R}$ such that ${}_0\alpha=1$. Then it is easy to see that A is a subgroup of the multiplicative semigroup of $\boldsymbol{R}$ (see e.g. W. Magnus [6], A. I. Mal'cev [9]).

1.2. Put

$$X_i = gp(1 + x_{i,\lambda} \mid \lambda \in \Lambda) \qquad (i = 1, 2);$$

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then it is clear that X_i is a free abelian group freely generated by the elements $1+x_{i,\lambda}$ $(\lambda\in\Lambda)$. The main result of this note is

THEOREM 1. *The subgroup of A generated by X_1 and X_2 is the (standard) wreath product of X_1 by X_2.*

It follows then immediately from Theorem 1 and a theorem of W. Magnus [7] that

COROLLARY 1. *The subgroup M generated by the elements*

$$a_\lambda = (1 + x_{2,\lambda})(1 + x_{1,\lambda}) \ (= 1 + x_{2,\lambda} + x_{1,\lambda}) \qquad (\lambda \in \Lambda)$$

is a free metabelian group freely generated by the elements a_λ $(\lambda\in\Lambda)$.

There is a second corollary of Theorem 1(see §3) which is worth mentioning, namely

COROLLARY 2. *Let E and F be torsion-free abelian groups. Then for a suitable choice of Λ, there is a monomorphism of E wr F into A (where E wr F denotes the (standard) wreath product of E by F).*

We have already remarked that $\bigcap_{i=1}^{\infty} R_i=0$ (see (3)). Consequently (see e.g. W. Magnus [6], A. I. Mal′cev [9]) the *wreath product E wr F of an arbitrary pair of torsion-free abelian groups E and F is residually torsion-free nilpotent.* This result is contained as a special case of a more general theorem due jointly to K. W. Gruenberg and the author [2].

1.3. Corollary 1 bears a striking resemblance to W. Magnus' original representation of a free group in a power series ring [6]. This remark is substantiated by the following analogue of W. Magnus' theorem that the so-called dimension subgroups and the lower central series of a free group coincide (see [6]).

THEOREM 2. *Let $M(i)$ be the set of those elements μ in M such that*

$$\text{order of } \mu - 1 \geqq i.$$

Then

$$M(i) = M_i,$$

where M_i is the ith term of the lower central series of M (note $M_1=M$).

Theorem 2 has already proved useful in [1].

1.4. Finally, we would like to remark that K. T. Chen [3] has already provided us with an explicit representation of a free metabelian group of rank two, and W. Magnus [7] has obtained a representation of any free metabelian group in a group of 2×2 matrices over a commutative ring.

None of these representations seem quite so amenable as that given by Corollary 1. Indeed it seems quite possible that much of the theory that is based on Magnus' representation [6] can be successfully mimicked (see for example W. Magnus [8]).

2. **The proof of Theorem 1.**

2.1. The proof that the subgroup generated by X_1 and X_2 is $X_1 \operatorname{wr} X_2$, depends on the form of the conjugates of the elements of X_1 by the elements of X_2. We begin by verifying that these conjugates commute—this is of course essential if X_1 and X_2 are to generate $X_1 \operatorname{wr} X_2$.

LEMMA 1. *The conjugates of elements in X_1 by elements in X_2 commute.*

PROOF. Let $1+\mu \in X_1$, $1+\tau \in X_2$. Notice that μ involves only $x_{1\lambda}$'s and τ only $x_{2\lambda}$'s. Then, using (1), we have

$$\begin{aligned}(1+\tau)^{-1}(1+\mu)(1+\tau) &= 1+(1+\tau)^{-1}\mu(1+\tau)\\ &= 1+(1-\tau+\tau^2-\cdots)\mu(1+\tau)\\ &= 1+\mu(1+\tau).\end{aligned} \tag{4}$$

Let, further, $1+\mu' \in X_1$, $1+\tau' \in X_2$. Then, by (1) and (4),

$$\begin{aligned}&(1+\tau)^{-1}(1+\mu)(1+\tau)(1+\tau')^{-1}(1+\mu')(1+\tau')\\ &\qquad= [1+\mu(1+\tau)][1+\mu'(1+\tau')]\\ &\qquad= 1+\mu(1+\tau)+\mu'(1+\tau')+\mu\mu'.\end{aligned} \tag{5}$$

It follows immediately from (1) and (5) that $(1+\tau)^{-1}(1+\mu)(1+\tau)$ and $(1+\tau')^{-1}(1+\mu')(1+\tau')$ commute.

2.2. In order now to complete the proof that $gp(X_1 \cup X_2) = X_1 \operatorname{wr} X_2$, we have to show that if

$$1+\tau_1,\ 1+\tau_2,\ \cdots,\ 1+\tau_k$$

are distinct elements of X_2, and if

$$1+\mu_1,\ 1+\mu_2,\ \cdots,\ 1+\mu_k$$

are elements of X_1 with $1+\mu_i \neq 1$, then

$$(1+\tau_1)^{-1}(1+\mu_1)(1+\tau_1)\cdots(1+\tau_k)^{-1}(1+\mu_k)(1+\tau_k) \neq 1. \tag{6}$$

Let us put w equal to the left-hand-side of (6). Then, by (4), we have

$$w = [1+\mu_1(1+\tau_1)][1+\mu_2(1+\tau_2)]\cdots[1+\mu_k(1+\tau_k)]. \tag{7}$$

We repeat that the μ_i involve only $x_{1\lambda}$'s and the τ_i only $x_{2\lambda}$'s. Suppose now that

(8) $$w = 1;$$

then it follows that

(9) $$\mu_1\tau_1 + \mu_2\tau_2 + \cdots + \mu_k\tau_k = 0.$$

Notice that every element $1+\tau \in X_2$ can be uniquely represented in the form

(10) $$1 + \tau = (1 + x_{2,\lambda_1})^{n_1} \cdots (1 + x_{2,\lambda_t})^{n_t} \qquad (\lambda_1 < \lambda_2 < \cdots < \lambda_t, n_j \neq 0).$$

By transforming both sides of (8) by suitably chosen elements of X_2 we can ensure that the elements $1+\tau_i$ occurring in (7) are of the form (cf. (10))

(11) $$1 + \tau_i = (1 + x_{2,\lambda(i,1)})^{n_{i,1}} \cdots (1 + x_{2,\lambda(i,l(1))})^{n_{i,l(i)}} \qquad (i = 1, 2, \cdots, k),$$

where $\lambda(i, 1) < \cdots < \lambda(i, l(i))$ and

(12) $$n_{i,1} > 0, \cdots, n_{i,l(i)} > 0 \qquad (i = 1, 2, \cdots, k)$$

(cf. (10)). As none of the elements $1+\tau_i$ coincide, it follows from (11) and (12) that they are uniquely determined by their *leading terms*

$$x_{2,\lambda(i,1)}^{n_{i,1}} \cdots x_{2,\lambda,i,(l(i))}^{n_{i,l(i)}},$$

i.e. these leading terms are all distinct. Bearing this in mind and observing that the order of $\mu_1 - 1$ is 1 i.e., ${}_1(\mu_1) \neq 0$, it follows immediately that there is no other term of degree

$$n_{1,1} + \cdots + n_{1,l(1)} + 1$$

on the left-hand side of (9) which matches

$${}_1(\mu_1) x_{2,\lambda(1,1)}^{n_{1,1}} \cdots x_{2,\lambda(1,l(1))}^{n_{1,l(1)}}.$$

So

$$\mu_1\tau_1 + \cdots + \mu_k\tau_k \neq 0.$$

This stands in direct conflict with (9). Hence the assumption (8) is invalid and (6) is indeed in force, as desired. This completes the proof of Theorem 1.

3. The proof of Corollary 2.

3.1. Let E and F be torsion-free abelian groups and let $\bar{E}$ and $\bar{F}$ be their completions i.e. minimal torsion-free divisible groups containing E and F respectively (see e.g. A. G. Kuroš [4, p. 165]). Notice that a torsion-free divisible group is a direct sum of copies of the additive group of rational numbers; the number of copies of the rationals involved is called the *rank* of the torsion-free abelian group.

Now choose a set Λ whose cardinality is the maximum of the ranks of $\overline{E}$ and $\overline{F}$. Let A be the subgroup of $\boldsymbol{R}$ introduced in 1.1, where the associated Λ is the one chosen above. We shall show that $\overline{E}$ wr $\overline{F}$ can be embedded in A. Since the subgroup of $\overline{E}$ wr $\overline{F}$ generated by E and F is simply E wr F we will have achieved the desired embedding.

3.2. We begin by observing that A is in fact a $\mathfrak{D}$-group i.e. a group in which every element has a unique nth root for every integer n ($\neq 0$); the nth root of an element can simply be computed component by component in view of (3). It follows easily that X_1 and X_2 can be embedded in minimal divisible (torsion-free abelian) subgroups $\overline{X}_1$ and $\overline{X}_2$ of A. If we denote by $a^{1/n}$ the nth root of an element $a \in A$, then $\overline{X}_i$ is simply generated by the elements $(1+x_{i,\lambda})^{1/n}$ for varying n and λ ($i=1, 2$). It is clear that $\overline{E}$ and $\overline{F}$ can be embedded in $\overline{X}_1$ and $\overline{X}_2$ respectively. So it remains to prove that $\overline{X}_1$ and $\overline{X}_2$ generate $\overline{X}_1$ wr $\overline{X}_2$.

3.3. We have only to verify that if

$$1 + \mu_1, \cdots, 1 + \mu_k \in \overline{X}_1 \quad \text{and} \quad 1 + \tau_1, \cdots, 1 + \tau_k \in \overline{X}_2$$

where the $1+\mu_i$ are all different from 1 and the $1+\tau_i$ are all different, then

$$(1 + \tau_1)^{-1}(1 + \mu_1)(1 + \tau_1) \cdot \cdots \cdot (1 + \tau_k)^{-1}(1 + \mu_k)(1 + \tau_k) \neq 1. \tag{13}$$

The mapping

$$\phi: x_{i,\lambda} \to (1 + x_{i,\lambda})^N - 1 \qquad (i = 1, 2)$$

defines an endomorphism, again denoted by ϕ, of $\boldsymbol{R}$ since the images of the $x_{i,\lambda}$ satisfy the relations (1). By choosing N suitably we can make sure that the elements $(1+\tau_j)\phi$ are distinct elements of X_2 and that $(1+\mu_j)\phi$ are nonunit elements of X_1 ($j=1, 2, \cdots, k$). But Theorem 1 now applies and so

$$[(1 + \tau_1)^{-1}\phi][(1 + \mu_1)\phi][(1 + \tau_1)\phi] \cdots \cdot [(1 + \tau_k)^{-1}\phi][(1 + \mu_k)\phi][(1 + \tau_k)\phi] \neq 1.$$

This establishes (13) and therefore also Corollary 2.

4. The proof of Theorem 2.

4.1. We may assume without loss of generality that Λ is finite, say

$$\Lambda = \{1, 2, \cdots, n\}$$

with the natural order.

Let c be an integer greater than 1. We are interested in the left-normed basic commutators in $a_1, \cdots, a_n$ of weight c.

LEMMA 2. *The left-normed basic commutators*[2] *of weight c freely generate, modulo R_{c+1}, a free abelian group.*

PROOF. Every left-normed basic commutator of weight c is of the form[3]

$$b = [a_{i_1}, a_{i_2}, \cdots, a_{i_c}] \tag{14}$$

where $1 \leqq i_j \leqq n$ and

$$i_1 > i_2, \; i_2 < i_3 < \cdots < i_c. \tag{15}$$

It is an easy matter to work out the leading term of b. Indeed

$$b = 1 + \{x_{1i_1}x_{2i_2} - x_{1i_2}x_{2i_1}\} x_{2,i_3} \cdots x_{2,i_c} + \text{terms of higher degree.} \tag{16}$$

It follows on inspecting (15) and (16) that these basic commutators (14) freely generate, modulo R_{c+1}, a free abelian group.

4.2. The proof of Theorem 2 is now easy.

First of all it is clear that $M_k \leqq M(k)$ for all k (e.g. from (16)).

Suppose then that $M_r = M(r)$. If $a \in M(r+1)$, then $a \in M_r$. So we can write $a = bc$, where b is a word in basic commutators of weight r and $c \in M_{r+1}$. If b is nontrivial, then by Lemma 2 the order of b is r; therefore the order of bc is also r. But this means that $a \notin M(r+1)$, a contradiction. Therefore $b = 1$ and so $a \in M_{r+1}$. Thus $M(r+1) \leqq M_{r+1}$. Putting this together with the inequality at the start of the proof yields the desired result.

4.3. In conclusion it is worth pointing out that Lemma 2 provides a proof of an unpublished theorem of W. Magnus, namely the independence of the left normed basic commutators in a free metabelian group (cf. Hanna Neumann [**10**]).

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[2] See P. Hall [**5**].

[3] Here $[x_1, x_2] = x_1^{-1}x_2^{-1}x_1x_2$ and inductively $[x_1, \cdots, x_n] = [[x_1, \cdots, x_{n-1}], x_n]$ for $n \geqq 2$.

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