OPEN MANIFOLDS WITH MONOTONE UNION PROPERTY

KYUNG WHAN KWUN

A space X has the *monotone union property* if whenever Y is the union of open sets $U_1 \subset U_2 \subset \cdots \subset Y$ with each U_i homeomorphic to X then Y is necessarily homeomorphic to X. Brown has shown [2] that the open n-cell has this property. The question arises as to which open manifolds have this property. In particular, one may ask if the open n-cell is, at least for high dimensions, characterized by this property among open n-manifolds. We show that there are many manifolds with this property. Namely we prove

THEOREM. *Let X be a closed p.l. manifold of* dim $n \neq 4$ *and $p \in X$. Then $X-p$ has the monotone union property.*

Simple examples show that contractible open manifolds or the interior of a compact manifold with 1-connected boundary do not have this property in general.

We thank P. Doyle for calling this problem to our attention.

We use $\mathring{M}$, $\dot{M}$ to denote the interior and the boundary of a topological manifold M. We divide the proof according to cases.

1. **For $n \geqq 5$ and X orientable.** Let $U_1 \subset U_2 \subset \cdots$ be a sequence with each U_i homeomorphic to $X-p$. We suppose that X is orientable and show that $U = \bigcup_i U_i$ is homeomorphic to $X-p$. Let A be a compact submanifold of X such that $[X-A]^-$ is a ball neighborhood of p in X.

Let, for each $i = 1, 2, \cdots, C_{i1} \subset C_{i2} \subset \cdots$ be a sequence of compact sets such that the union is U_i. Define $C_j = C_{1j} \cup C_{2j} \cup \cdots \cup C_{jj}$. Then $C_j \subset U_j$ and $\bigcup_j C_j = U$.

Let h_i be homeomorphisms of $X-p$ onto U_i and $A_i = h_i(A)$. By an inductive choice of h_i, we suppose that $\mathring{A}_i \supset C_i \cup A_{i-1}$ and therefore $\bigcup A_i = U$. Now we show that $\mathring{A}_i - \mathring{A}_{i-1}$ is homeomorphic to $S^{n-1} \times [0, 1)$ for each i. Now fix i and view U_i as having the $p.l.$ structure induced by h_i and $X-p$. Let $W = A_i - \mathring{A}_{i-1}$.

(1.1) $\pi_1 W = 1$. To see this, consider $\pi_1 A_{i-1} = \pi_1 A_i = \pi_1 A_{i-1} * \pi_1 W$.

By Grusko's theorem [3], $\pi_1 W = 1$.

(1.2) $\qquad \dot{A}_i \subset W$ is a homotopy equivalence.

Received by the editors March 30, 1966.

Since W and $\dot{A}_i$ are 1-connected by 1.1, by Whitehead's theorem, it suffices to show that it induces homology isomorphisms or $H_*(W, \dot{A}_i) = 0$. This is equivalent to showing $H_*(W, \dot{A}_{i-1}) = 0$ by the relative Poincaré duality. Then by excision, we need only show $H_*(A_i, A_{i-1}) = 0$, or $A_{i-1} \subset A_i$ induces homology isomorphisms. Consider a part of the Mayer-Vietoris sequence for the triad $(A_i; A_{i-1}, W)$, namely

$$0 \to H_{n-1}(\dot{A}_{i-1}) \xrightarrow{\phi} H_{n-1}(A_{i-1}) \oplus H_{n-1}(W) \xrightarrow{\Psi} H_{n-1}(A_i) \to 0.$$

Since $\phi(H_{n-1}(\dot{A}_{i-1})) \subset H_{n-1}(W)$ as $\dot{A}_{i-1}$ bounds A_{i-1},

$$H_{n-1}(A_{i-1}) \oplus (H_{n-1}(W)/\mathrm{Im}\,\phi) \overset{\Psi'}{\simeq} H_{n-1}(A_i).$$

Since $H_{n-1}(A_{i-1}) \approx H_{n-1}(A_i)$ is free abelian, $H_{n-1}(W)/\mathrm{Im}\phi = 0$ and $A_{i-1} \subset A_i$ induces an isomorphism $H_{n-1}(A_{i-1}) \simeq H_{n-1}(A_i)$. That $A_{i-1} \subset A_i$ induces isomorphisms for other dimensions is simple.

(1.3) $A_i - A_{i-1}$ is homeomorphic to $S^{n-1} \times [0, 1)$.

Now $A_i - A_{i-1}$ is a *p.l.* submanifold of $U_i = h_i(X-p)$. Furthermore, $\dot{A}_i$ is a *p.l.* sphere. Attach a combinatorial n-ball B along $\dot{A}_i$ to obtain a *p.l.* manifold $V = B \cup (A_i - A_{i-1})$. Then $H_*(V, B) \simeq H_*(A_i - A_{i-1}, \dot{A}_i) = H_*(W, \dot{A}_i) = 0$ by 1.2. Hence $B \subset V$ is a homotopy equivalence or V is contractible. Since V is 1-connected at infinity, by [5], V is n-space. Now by the generalized Schoenflies theorem [1], $A_i - A_{i-1}$ is homeomorphic to $S^{n-1} \times [0, 1)$.

(1.4) $\mathring{A}_i - \mathring{A}_{i-1}$ is homeomorphic to $S^{n-1} \times [0, 1)$.

This follows from (1.3) by a simple argument.

(1.5) U is homeomorphic to $X - p$.

This can be directly proved. But in order to avoid epsilontics, we use [2]. Replace A_1 by n-cell. So that the situation is the monotone union of open n-cells. In this case $U - \mathring{A}_1$ is homeomorphic to $S^{n-1} \times [0, 1)$. So that $\mathring{A}_1$ is homeomorphic to U.

2. **For $n \geqq 5$ and X nonorientable.** Now suppose X is nonorientable. Then so is $X - p$. Consider the similar U_i, C_i, A_i. Consider the inclusion $A_{i-1} \subset A_i$. Let $p: B_i \to A_i$ be the 2-sheeted orientable covering of A_i. Let S_i, S_i', S_{i-1} and S_{i-1}' be the disjoint $(n-1)$-spheres in B_i such that $p^{-1}(\dot{A}_j) = S_j \cup S_j'$. One argues that $B_{i-1} = p^{-1}(A_{i-1})$ has two complementary domains P and P' in B_i such that $P \supset S_i$, $P' \supset S_i'$. For instance, as before $\pi_1(A_i - A_{i-1}) = 1$ by Van Kampen's theorem and Grusko's theorem, and $p^{-1}(A_i - \mathring{A}_{i-1})$ is the disjoint union of two

homeomorphic copies. Let $W=\overline{P}$, $W'=\overline{P}'$. Once we show that $S_{i-1}\subset W$ (therefore also $S'_{i-1}\subset W'$) induces homology isomorphisms, the result can be proved along a line similar to the orientable case. Consider the exact sequence

$$0 \to H_{n-1}(S_{i-1} \cup S'_{i-1}) \xrightarrow{\phi} H_{n-1}(B_{i-1}) \oplus H_{n-1}(W) \oplus H_{n-1}(W')$$
$$\xrightarrow{\Psi} H_{n-1}(B_i) \to 0$$

which is a part of the Mayer-Vietoris sequence for the triad (B_i; B_{i-1}, $W\cup W'$). Every group appearing above is a finitely generated free abelian group as B_i is compact and orientable. Hence $H_{n-1}(W) \simeq H_{n-1}(W') \simeq Z$. Let $b\in H_{n-1}(B_{i-1})$ be such that $(b, 0, 0) \in \operatorname{Ker} \Psi = \operatorname{Im}\Phi$. Since $H_{n-1}(S_{i-1})\to H_{n-1}(W')=Z$ and $H_{n-1}(S_{i-1})\to H_{n-1}(W)\simeq Z$ are nontrivial, $b=0$. On the other hand, cokernel $[H_{n-1}(B_{i-1})\to H_{n-1}(B_i)]$ is isomorphic to a torsion subgroup of $H_{n-1}(B_i, B_{i-1})\simeq H_{n-1}(W, S_{i-1})\oplus H_{n-1}(W', S'_{i-1})$ which is free. Hence $H_{n-1}(B_{i-1})\to H_{n-1}(B_i)$ is an isomorphism. This completes the proof except for homology isomorphisms at dimensions $\neq n-1$. But this is again easy to show.

3. **For** $n\leqq 3$. We only consider the case in which X is orientable. For nonorientable case one again passes to the orientable covering. Now the case $n=2$ is trivial, so we suppose X is orientable and of dimension 3. Let A_1, A_2 be compact manifolds homeomorphic to A such that $A_1\subset \mathring{A}_2$. It suffices to show that $A_2-\mathring{A}_1$ is homeomorphic to $S^2\times[0, 1]$. Let M_2 is a manifold obtained from A_2 by attaching 3-cell, M_1 obtained from A_1 by attaching a 3-cell. Let M_3 be a manifold obtained from $A_2-\mathring{A}_1$ by attaching two disjoint 3-cells. Then $M_2 = M_1\# M_3$. By [**4**], M_3 is a 3-sphere so that $A_2-\mathring{A}_1$ is homeomorphic to $S^2\times[0, 1]$.

References

1. Morton Brown, *A proof of the generalized Schönflies theorem*, Bull. Amer. Math. Soc. **66** (1960), 74–76.

2. ———, *The monotone union of open n-cells is an open n-cell*, Proc. Amer. Math. Soc. **12** (1961), 812–814.

3. A. G. Kurosh, *The theory of groups*, Vol. II, Chelsea, New York, 1956.

4. John Milnor, *A unique decomposition theorem for 3-manifolds*, Amer. J. Math. **84** (1962), 1–7.

5. John Stallings, *The piecewise-linear structure of Euclidean space*, Proc. Cambridge Philos. Soc. **58** (1962), 481–488.

Michigan State University