REMARKS ON THE BORDISM ALGEBRA OF INVOLUTIONS

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1. **Introduction.** Let $\mathfrak{N}_*$ denote the unoriented bordism ring and let $\mathfrak{N}_*(Z_2)$ be the unoriented bordism group of involutions. Then $\mathfrak{N}_*(Z_2)$ is a module over $\mathfrak{N}_*$. In [1], it is shown that $\mathfrak{N}_*(Z_2)$ has $\{[A, S^n]\}_{n=0}^{\infty}$ as a basis over $\mathfrak{N}_*$, where $[A, S^n]$ is the bordism class of the antipodal involution on the n-sphere. Let $x_n = [A, S^n] + \sum_{j=0}^{n-1} [P^{n-j}] x_j$ for each $n \geqq 0$. Then $\{x_n\}_{n=0}^{\infty}$ is also a basis for $\mathfrak{N}_*(Z_2)$ over $\mathfrak{N}_*$. This has the advantage that x_n belongs to the reduced group $\tilde{\mathfrak{N}}_*(Z_2)$ for $n \geqq 1$.

In [2], Su showed that $\mathfrak{N}_*(Z_2)$ is a Hopf algebra over $\mathfrak{N}_*$, the multiplication being induced by the H-space multiplication of the classifying space $B(Z_2)$, and the comultiplication being induced by the diagonal map. We refer the reader to [1] and [2] for definitions and terminology.

It is natural then to ask for the multiplication law in $\mathfrak{N}_*(Z_2)$. In [2], Su showed that the multiplication satisfies $x_m x_n = (m, n) x_{m+n}$ mod A_{m+n} where $(m, n) = (m+n)!/m!n!$ and A_{m+n} is the $\mathfrak{N}_*$ module generated by $x_0, x_1, \cdots, x_{m+n-1}$. In general, this congruence cannot be replaced by an actual equation. For example, by explicit computation, one can show that $x_1 x_2 = x_3 + [P^2] x_1$. In this note, we show, however, that this congruence can be replaced by an equation if m and n are both odd. Precisely, we show

THEOREM.

$$x_{2m+1} x_{2n+1} = 0 \qquad \textit{for all } m, n.$$

We are unable to give a complete description of the multiplication although we have some partial results.

2. In this section, $[A, S^n]$ will always denote the bordism class of the antipodal involution on the n-sphere. In connection with any bordism class $[T, M^n]$, c will always stand for the characteristic class of the involution. Stiefel Whitney classes of manifolds will always be denoted by w_i. Also, the letter d, with or without subscripts, will always denote the generator of $H^1(P^n, Z_2)$, the dimension n of the projective space P^n being clear from the context. Finally, homology and cohomology will always be taken with coefficients mod 2.

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LEMMA 1. *$[A, S^1][A, S^n]=0$ for odd n.*

PROOF. The characteristic class of the involution is given by

$$c = d_1 \otimes 1 + 1 \otimes d_2.$$

Further, the Stiefel-Whitney classes of $P^1 \times P^n$ are given by

$$w_i = \binom{n+1}{i} 1 \otimes d_2^i.$$

Now $[A, S^1][A, S^n]$ is completely determined by its involution numbers

$$\langle w_{i_1} \cdots w_{i_r}(P^1 \times P^n)c^l, P^1 \times P^n \rangle$$

where $i_1 + \cdots + i_r = n+1-l$. But

$$c^l = \sum_{i=0}^{1} \binom{l}{i} d_1^i \otimes d_2^{l-i}.$$

Hence the above involution number is

$$\sum_{i=0}^{1} \binom{l}{i} \left\langle \prod_{j=1}^{r} \binom{n+1}{i_j} d_1^{i_j} \otimes d_2^{n+1-i_j}, P^1 \times P^n \right\rangle = l \prod_{j=1}^{r} \binom{n+1}{i_j}.$$

This is zero if l is even. On the other hand, if l is odd, then the sum $i_1 + \cdots + i_r = n+1-l$ is odd since n is odd. Hence some i_j is odd. But this means that

$$\binom{n+1}{i_j} = 0.$$

Thus all the involution numbers vanish and hence the lemma follows.

NOTE. Since $x_1 = [A, S^1]$, we have then that $x_1[A, S^{2n+1}] = 0$ for all n.

LEMMA 2. *$x_1 x_{2n+1} = 0$ for all $n \geqq 0$.*

PROOF. If $n=0$, then $x_1^2 = 0$ by Proposition 3.4 of [2]. We now proceed by induction. Suppose that the result is true for all $j < n$. Now

$$x_1 x_{2n+1} = x_1[A, S^{2n+1}] + \sum_{j=0}^{2n} x_1 x_j [P^{2n+1-j}]$$

$$= \sum_{j=0}^{2n} x_1 x_j [P^{2n+1-j}] \text{ by Lemma 1.}$$

Since $[P^{2n+1-j}]=0$ for even j, we have that

$$x_1x_{2n+1} = \sum_{j=0}^{n-1} x_1x_{2j+1}[P^{2n-2j}].$$

By hypothesis $x_1x_{2j+1}=0$ for all $j<n$. Hence $x_1x_{2n+1}=0$.

LEMMA 3. *$x_1x_n=0$ if and only if n is odd.*

PROOF. If n is odd, the result follows by Lemma 2. It remains to show that if $x_1x_n=0$ then n is odd, or equivalently that if n is even then $x_1x_n\neq 0$. But by Su's result, we have that $x_1x_n=x_{n+1} \bmod A_{n+1}$ if n is even. Hence $x_1x_n\neq 0$.

THEOREM. *$x_{2n+1}\in x_1\mathfrak{R}_*(Z_2)$ for all n and hence $x_{2m+1}x_{2n+1}=0$ for all m, n.*

PROOF. We first prove that $x_{2n+1}\in x_1\mathfrak{R}_*(Z_2)$. Clearly this is true if $n=0$. We now proceed by induction. Suppose $x_{2i+1}\in x_1\mathfrak{R}_*(Z_2)$ for all $i<n$. Now

$$x_1x_{2n} = x_{2n+1} + \sum_{j=0}^{2n} a_{2n+1-j}x_j$$

where $a_{2n+1-j}\in\mathfrak{R}_{2n+1-j}$. Multiplying by x_1 and applying Lemma 2 we obtain the equation

$$\sum_{j=0}^{2n} a_{2n+1-j}x_1x_j = 0.$$

Again applying Lemma 2, we simplify the expression to obtain

$$\sum_{j=0}^{n} a_{2n+1-2j}x_1x_{2j} = 0.$$

Writing this out in detail we obtain

$$a_1x_1x_{2n} + a_3x_1x_{2n-2} + \cdots + a_{2n-1}x_1x_2 + a_{2n+1}x_1 = 0.$$

Now we recall that $x_1x_{2j}=x_{2j+1}+y_j$ where $y_j\in A_{2j+1}$. Hence we obtain the equation

$$a_1x_{2n+1} + a_1y_n + a_3x_{2n-1} + a_3y_{n-1} + \cdots + a_{2n-1}x_3 + a_{2n-1}y_1 + a_{2n+1}x_1 = 0.$$

Since the x_j form a basis, we conclude that

$$0 = a_1 = a_3 = \cdots = a_{2n-1} = a_{2n+1}.$$

Thus we now have

$$x_1 x_{2n} = x_{2n+1} + \sum_{j=0}^{n-1} a_{2n-2j} x_{2j+1}.$$

By hypothesis, each $x_{2j+1} \in x_1 \mathfrak{R}_*(Z_2)$ for $j \leqq n-1$. Hence $x_{2n+1} \in x_1 \mathfrak{R}_*(Z_2)$. Finally, the last statement of the theorem follows from the fact that the multiplication is commutative and that $x_1^2 = 0$.

References

1. P. E. Conner and E. E. Floyd, *Differentiable periodic maps*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Vol. 33, Springer-Verlag, Berlin, 1964.

2. J. C. Su, *A note on the bordism algebra of involutions*, Michigan Math. J. **12** (1965), 25–31.

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