CHARACTERISTIC ROOTS OF M-MATRICES

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A square matrix B is called a nonnegative matrix (written $B \geqq 0$) if each element of B is a nonnegative number. It is well known [1] that every nonnegative matrix B has a nonnegative characteristic root $p(B)$ (the *Perron root of* B) such that each characteristic root λ of B satisfies $|\lambda| \leqq p(B)$.

A square matrix A is called an *M-matrix* if it has the form $k \cdot I - B$, where B is a nonnegative matrix, $k > p(B)$, and I denotes the identity matrix. In case A is a real, square matrix with nonpositive off-diagonal elements, each of the following is a necessary and sufficient condition for A to be an M-matrix [4, p. 387].

(1) Each principal minor of A is positive.

(2) A is nonsingular, and $A^{-1} \geqq 0$.

(3) Each real characteristic root of A is positive.

(4) There is a row vector x with positive entries ($x > 0$) such that $xA > 0$.

If A is an M-matrix, then A has a positive characteristic root $q(A)$ which is *minimal*, in the sense that for each characteristic root β of A, $q(A) \leqq |\beta|$ [4, p. 389]. Bounds for $q(A)$ can be readily obtained using known bounds for the Perron root of a nonnegative matrix. In this paper, as in [2], we reverse this procedure, studying the characteristic roots of M-matrices in order to find new bounds for Perron roots.

Ky Fan proved the following lemma [3]:

LEMMA A. *Let* $A = (a_{ij})$ *be an* M*-matrix of order* n. *Then the matrix* $C = (c_{ij})$ *given by*

$$c_{ij} = a_{ij} - a_{in}a_{nj}(1/a_{nn}) \qquad (i, j = 1, 2, \cdots, n-1).$$

is an M*-matrix, and* $c_{ij} \leqq a_{ij}$ $(i, j = 1, 2, \cdots, n-1)$.

In [2] we generalized Lemma A by using certain principal minors in place of single elements of A. Here we prove a different generalization of Fan's lemma, contained in Theorem 1.

We denote the submatrix of a matrix A formed using rows $i_1, i_2, \cdots, i_p$ and columns $j_1, j_2, \cdots j_p$ by $A(i_1, i_2, \cdots, i_p; j_1, j_2, \cdots, j_p)$. For principal submatrices we abbreviate this to $A(i_1, i_2, \cdots, i_p)$. $A_{i,j}$ denotes the element of A in row i and column j.

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THEOREM 1. *Let M be an M-matrix of order $n=mk$, partitioned into the form*

$$M=\begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1k} \\ A_{21} & A_{22} & \cdots & A_{2k} \\ \cdot & \cdot & \cdots & \cdot \\ A_{k1} & A_{k2} & \cdots & A_{kk} \end{bmatrix},$$

where each A_{ij} is an $m\times m$ matrix. Let $\phi(M)$ denote the matrix

$$\begin{bmatrix} B_{11} & B_{12} & \cdots & B_{1,k-1} \\ B_{21} & B_{22} & \cdots & B_{2,k-1} \\ \cdot & \cdot & \cdots & \cdot \\ B_{k-1,1} & B_{k-1,2} & \cdots & B_{k-1,k-1} \end{bmatrix},$$

where $B_{ij}=A_{ij}-A_{ik}A_{kk}^{-1}A_{kj}$. Then

(5) *$\phi(M)$ is an M-matrix, of order $m(k-1)$,*

(6) *$\phi(M)_{i,j}\leqq M_{i,j}$, for $i, j=1, 2, \cdots, m(k-1)$,*

(7) *$\det\phi(M)=\det M/\det A_{kk}$, and*

(8) *$q(\phi(M))\geqq q(M)$.*

The proof of the theorem depends on two lemmas.

LEMMA 1. *Let $T=(t_{ij})$ be an M-matrix of order n, with minimal characteristic root $q(T)$. Let λ be a number such that $\lambda<q(T)$. Then* $\det(T-\lambda I)>0$.

PROOF. If $\omega_1, \omega_2, \cdots, \omega_n$ are the characteristic roots of T, $\det(T-\lambda I) = \prod_{i=1}^{n}(\omega_i-\lambda)$. For each real ω_i, $\omega_i\geqq q(T)>\lambda$, so that $\omega_i-\lambda>0$. The complex factors $\omega_i-\lambda$ occur in conjugate pairs whose product is positive. Thus $\det(T-\lambda I)>0$.

LEMMA 2. *Let $S=(s_{ij})$ be an M-matrix of order n, let λ be a number such that $0<\lambda<q(S)$, and let t be an integer such that $0\leqq t<n$. Then the determinant of the matrix obtained from S by subtracting λ from each of the first t main diagonal elements $s_{11}, s_{22}, \cdots, s_{tt}$ is positive.*

PROOF. Let T be the matrix obtained from S by adding λ to the main diagonal elements $s_{t+1,t+1}, \cdots, s_{n,n}$. Since $\lambda>0$, $T\geqq S$ and the off-diagonal elements of T are nonpositive, so [**4**, p. 389] T is an M-matrix and $q(T)\geqq q(S)$. Thus, since $\lambda<q(T)$, Lemma 1 implies that $\det(T-\lambda I)>0$, which proves Lemma 2.

PROOF OF THEOREM 1. Since M has nonpositive off-diagonal elements, A_{ik} and A_{kj} are nonpositive matrices if $i, j\neq k$. Also A_{kk}, a principal submatrix of M, is itself an M-matrix [**4**, p. 390], so that

$A_{kk}^{-1} \geqq 0$. Thus $A_{ik}A_{kk}^{-1}A_{kj} \geqq 0$, so that $B_{ij} \leqq A_{ij}$, verifying (6). So $\phi(M)$ has nonpositive off-diagonal elements, and to show that $\phi(M)$ is an M-matrix we prove now that each principal minor of $\phi(M)$ is positive.

Let F be the nonnegative, $n \times n$ matrix

$$F = \begin{bmatrix} I & 0 \\ 0 & A_{kk}^{-1} \end{bmatrix}.$$

Then

$$MF = \begin{bmatrix} A_{11} & \cdots & A_{1k}A_{kk}^{-1} \\ A_{21} & \cdots & A_{2k}A_{kk}^{-1} \\ \cdot & \cdots & \cdot \\ A_{k1} & \cdots & I \end{bmatrix}$$

has nonpositive off-diagonal elements. Also, by (4), since M is an M-matrix, there is a vector x with positive entries ($x > 0$) such that $xM > 0$. Then also $(xM)F > 0$, since $F \geqq 0$ and F has at least one positive element in each column. (Each main diagonal element of the inverse of an M-matrix, being the quotient of two positive principal minors, is positive.) Thus, by (4), MF is an M-matrix. Moreover, $\phi(M) = \phi(MF)$, so it suffices to show that each principal minor of $\phi(MF)$ is positive.

To this end, write

$$MF = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1k} \\ C_{21} & C_{22} & \cdots & C_{2k} \\ \cdot & \cdot & \cdots & \cdot \\ C_{k1} & C_{k2} & \cdots & I \end{bmatrix},$$

with $C_{ij} = A_{ij}$ if $j \neq k$, and $C_{ik} = A_{ik}A_{kk}^{-1}$. Then it is easily verified that a typical element of $\phi(MF)$, say $(C_{ij} - C_{ik}C_{kj})_{\alpha,\beta}$, can be written as the determinant

$$\det \begin{bmatrix} (C_{ij})_{\alpha,\beta} & (C_{ik})_{\alpha,1} & (C_{ik})_{\alpha,2} & \cdots & (C_{ik})_{\alpha,m} \\ (C_{kj})_{1,\beta} & & & & \\ (C_{kj})_{2,\beta} & & I & & \\ \vdots & & & & \\ (C_{kj})_{m,\beta} & & & & \end{bmatrix}.$$

Thus by Sylvester's identity [**5**, p. 16], if $\phi(MF)(i_1, i_2, \cdots, i_t)$ is a principal submatrix of $\phi(MF)$, then

$$\det\{\phi(MF)(i_1, i_2, \cdots, i_t)\}$$
$$= (\det I)^{t-1} \det (MF)(i_1, i_2, \cdots, i_t, n-m+1, \cdots, n) > 0.$$

This completes the proof that $\phi(M)$ is an M-matrix.

To verify (7), we use Sylvester's identity again to write

$$\det(\phi(M)) = \det \phi(MF) = \det MF = \det M \det F$$
$$= \det M/\det A_{kk}.$$

Finally, to show that $q(\phi(M)) \geqq q(M)$, we write the characteristic polynomial of $\phi(M)$ in the form

$$\det(\phi(M) - xI) = \det \begin{bmatrix} D_{11} & \cdots & D_{1,k-1} \\ \cdot & \cdots & \cdot \\ D_{k-1,1} & \cdots & D_{k-1,k-1} \end{bmatrix},$$

where $D_{ij} = (A_{ij} - \delta_{ij}xI) - A_{ik}A_{kk}^{-1}A_{kj}$. ($\delta_{ij}$ is the Kronecker symbol.) Then clearly $\phi(M) - xI$ is the same as the matrix $\phi(Q(x))$, where $Q(x)$ agrees with the matrix M except that the matrix xI has been subtracted from each of the submatrices $A_{11}, A_{22}, \cdots, A_{k-1,k-1}$. (The construction of $\phi(Q(x))$ from $Q(x)$, and part (7) of Theorem 1, require only that A_{kk}^{-1} exists, and not that $Q(x)$ be an M-matrix.) So as in (7), we have $\det(\phi(M) - xI) = \det \phi(Q(x)) = \det Q(x)/\det A_{kk}$.

Now if $0 < \lambda < q(M)$, Lemma 2 implies that $\det Q(\lambda) > 0$. Also, $\det A_{kk} > 0$, since it is a principal minor of M. Thus $\det(\phi(M) - \lambda I) > 0$, and so no real characteristic root of $\phi(M)$ is less than $q(M)$. I.e. $q(\phi(M)) \geqq q(M)$, completing our proof.

Applying Theorem 1, we obtain the following theorem about the Perron root of a nonnegative matrix.

THEOREM 2. *If B is a nonnegative matrix of order $n = mk$, and $h > p(B)$, the Perron root of B, then*

(9) *$D = hI - \phi(hI - B)$ is a nonnegative matrix of order $m(k-1)$,*

(10) *$D_{ij} \geqq B_{ij}$ for $i, j = 1, 2, \cdots, m(k-1)$, and*

(11) *$p(B) \geqq p(D)$.*

PROOF. Let $M = hI - B$. Then $D_{ij} = h\delta_{ij} - \phi(M)_{ij} \geqq h\delta_{ij} - M_{ij} = B_{ij} \geqq 0$, verifying (9) and (10). Also $h - p(D) = q(\phi(M)) \geqq q(M) = h - p(B)$, so that $p(B) \geqq p(D)$.

As in [2], the use of known lower bounds for the Perron root of D leads to bounds for $p(B)$. Similarly, Theorem 1 gives new bounds for $q(M)$, by applying known upper bounds for the minimal characteristic root of the M-matrix $\phi(M)$.

Added in proof. The author has since discovered the fact that the matrix $\phi(M)$ can also be constructed using the method in [**2**, Lemma 1].

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