SUMS OF COUNTABLE PRIMARY GROUPS

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This note contains, in particular, a short proof of the celebrated result of G. Kolettis which states that a group G is determined by its Ulm invariants if G is the direct sum of countable, reduced, p-primary groups [1]. All groups considered are commutative.

THEOREM. *Suppose that* $G=\sum_{\lambda\in\Lambda}A_\lambda$ *and* $H=\sum_{\lambda\in\Lambda}B_\lambda$ *are decompositions of G and H into direct sums of countable, reduced, p-groups. If G and H have the same Ulm invariants, then there exists a partition of* Λ *into countable subsets* Λ_μ, $\mu\in M$, *such that* $G_\mu=\sum_{\lambda\in\Lambda_\mu}A_\lambda$ *is isomorphic to* $H_\mu=\sum_{\lambda\in\Lambda_\mu}B_\lambda$ *for each* $\mu\in M$.

PROOF. The theorem is vacuous if Λ is countable, so assume that Λ is uncountable and let Ω be the smallest ordinal having the cardinality of the set Λ. Let $G[p]=\sum_{i\in I}\{x_i\}$. Since G and H are direct sums of countable groups and have the same Ulm invariants, there is[1] a height-preserving isomorphism π from $G[p]$ onto $H[p]$—an element of G is said to have height α if it is contained in $p^\alpha G$ but not in $p^{\alpha+1}G$. Thus corresponding to the decomposition $G[p]=\sum_{i\in I}\{x_i\}$ is the decomposition $H[p]=\sum_{i\in I}\{y_i\}$ where $y_i=\pi(x_i)$. Since $|G[p]| =|G|=|\Lambda|=|\Omega|$, the index set I can be taken as the initial segment of ordinals less than Ω.

Suppose that $\gamma<\Omega$ and that for each $\beta<\gamma$ we have shown the existence of a subset S_β of Λ and a subset I_β of I such that the following conditions are satisfied.

(1) $$\sum_{\lambda\in S_\beta}A_\lambda[p]=\sum_{i\in I_\beta}\{x_i\} \quad\text{and}\quad \sum_{\lambda\in S_\beta}B_\lambda[p]=\sum_{i\in I_\beta}\{y_i\}.$$

(2) $$|S_\beta|=|I_\beta|\leqq\aleph_0|\beta|.$$

(3) $$\alpha\in I_\beta \text{ if } \alpha<\beta.$$

(4) $$S_\beta=\bigcup_{\alpha<\beta}S_\alpha \quad\text{and}\quad I_\beta=\bigcup_{\alpha<\beta}I_\alpha \text{ if } \beta \text{ is a limit ordinal.}$$

(5) $$S_\alpha\subseteq S_\beta \quad\text{and}\quad I_\alpha\subseteq I_\beta \text{ if } \alpha<\beta.$$

If γ is a limit ordinal, we define $S_\gamma=\bigcup_{\alpha<\gamma}S_\alpha$ and $I_\gamma=\bigcup_{\alpha<\gamma}I_\alpha$ and observe that the conditions (1)–(5) remain valid for $\beta\leqq\gamma$.

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[1] If G is a countable, reduced, primary group, then $G[p]$ can be decomposed as $G[p]=\Sigma S_\alpha$ where the elements of S_α have height α. The result then easily extends to direct sums of countable groups.

Suppose that γ is not a limit ordinal. There is a minimal subset $S_{\gamma,1}$ of Λ such that $\{\sum_{S_{\gamma-1}} A_\gamma, x_{\gamma-1}\} \subseteq \sum_{S_{\gamma,1}} A_\lambda$ and $\{\sum_{S_{\gamma-1}} B_\lambda, y_{\gamma-1}\} \subseteq \sum_{S_{\gamma,1}} B_\lambda$. There is a minimal subset $I_{\gamma,1}$ of I such that $\sum_{S_{\gamma,1}} A_\lambda[p] \subseteq \sum_{I_{\gamma,1}}\{x_i\}$ and $\sum_{S_{\gamma,1}} B_\lambda[p] \subseteq \sum_{I_{\gamma,1}}\{y_i\}$. Then there is a minimal set $S_{\gamma,2}$ ($\supseteq S_{\gamma,1}$) of Λ such that $\sum_{I_{\gamma,1}}\{x_i\} \subseteq \sum_{S_{\gamma,2}} A_\lambda$ and $\sum_{I_{\gamma,1}}\{y_i\} \subseteq \sum_{S_{\gamma,2}} B_\lambda$. Continuing in this way, we obtain ascending sequences $S_{\gamma,n}$ and $I_{\gamma,n}$ such that

$$\sum_{S_{\gamma,n}} A_\lambda[p] \subseteq \sum_{I_{\gamma,n}} \{x_i\} \subseteq \sum_{S_{\gamma,n+1}} A_\lambda[p]$$

and

$$\sum_{S_{\gamma,n}} B_\lambda[p] \subseteq \sum_{I_{\gamma,n}} \{y_i\} \subseteq \sum_{S_{\gamma,n+1}} B_\lambda[p].$$

Note that $|S_{\gamma,n}| \leqq \aleph_0 |\gamma|$ for each positive integer n. Set $S_\gamma = \bigcup S_{\gamma,n}$ and $I_\gamma = \bigcup I_{\gamma,n}$. Then conditions (1)–(5) hold for $\beta \leqq \gamma$. Obviously we can use the scheme described above to show the existence of countably infinite sets S_1 and I_1 which satisfy conditions (1) and (2) when $\beta = 1$; hence there exist, for each $\beta < \Omega$, a subset S_β of Λ and a subset I_β of I satisfying conditions (1)–(5). Condition (3) implies that $I = \bigcup_{\beta<\Omega} I_\beta$ and condition (1) implies that $\Lambda = \bigcup_{\beta<\Omega} S_\beta$.

Define $\Lambda_0 = S_1$ and $\Lambda_\beta = S_{\beta+1} - S_\beta$ for $1 \leqq \beta < \Omega$. We know that $\sum_{\lambda\in\Lambda_0} A_\lambda$ and $\sum_{\lambda\in\Lambda_0} B_\lambda$ have the same Ulm invariants, for (the restriction of) π is a height-preserving isomorphism between their socles $\sum_{i\in I_1}\{x_i\}$ and $\sum_{i\in I_1}\{y_i\}$. We wish to show that $\sum_{\lambda\in\Lambda_\beta} A_\lambda$ and $\sum_{\lambda\in\Lambda_\beta} B_\lambda$ have the same Ulm invariants. Since

$$\sum_{S_{\beta+1}} A_\lambda = \sum_{S_\beta} A_\lambda + \sum_{\Lambda_\beta} A_\lambda$$

and

$$\sum_{S_{\beta+1}} B_\lambda = \sum_{S_\beta} B_\lambda + \sum_{\Lambda_\beta} B_\lambda$$

and since the height-preserving isomorphism π from $\sum_{S_{\beta+1}} A_\lambda[p]$ onto $\sum_{S_{\beta+1}} B_\lambda[p]$ maps $\sum_{S_\beta} A_\lambda[p]$ onto $\sum_{S_\beta} B_\lambda[p]$, the composition mapping $\phi\pi$ is a height-preserving isomorphism from $\sum_{\lambda\in\Lambda_\beta} A_\lambda[p]$ onto $\sum_{\lambda\in\Lambda_\beta} B_\lambda[p]$ where ϕ is the projection of $\sum_{S_{\beta+1}} B_\lambda$ onto $\sum_{\Lambda_\beta} B_\lambda$. Thus $\sum_{\lambda\in\Lambda_\beta} A_\lambda$ and $\sum_{\lambda\in\Lambda_\beta} B_\lambda$ have the same Ulm invariants.

Since $|\Lambda_\beta| \leqq \aleph_0 |\beta| < |\Lambda|$ for each $\beta < \Omega$, the theorem now follows by induction on $|\Lambda|$.

Reference

1. G. Kolettis, *Direct sums of countable groups*, Duke Math. J. **27** (1960), 111–125.

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