MAXIMAL SEMICHARACTERS

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In this note the following notations are used.

$G$ denotes a commutative semigroup with identity $1$ and $H$ a sub-
semigroup containing $1$.

$N$ is a real-valued function on $G$ such that (i) $N \geq 0$ (ii) $N(1) = 1$
(iii) $N(xy) \leq N(x)N(y)$ for elements $x, y$ in $G$.

A real homomorphism $\phi$ is a function on $G$ (or $H$) satisfying (i), (ii)
and $\phi(xy) = \phi(x)\phi(y)$.

Our interest is in real homomorphisms subject to the inequality
$N \geq \phi \geq 0$, and especially in homomorphisms of this type which are
maximal in that $N \geq \sigma \geq \phi \geq 0$ implies $\phi = \sigma$.

**THEOREM 1.** Suppose that $\phi$ is a real homomorphism of $H$ and
(iv) $a\alpha_1 = b\alpha_2 (a, b \in G, \alpha_1 \in H)$ implies $N(a)\phi(h_1) \geq \phi(h_2)$.
Then $\phi$ can be extended to a real homomorphism $\sigma$ for which $N \geq \sigma \geq 1$.

**PROOF.** In the family $\{N'\}$ of functions $N' \leq N$ which fulfill re-
quirements (i)–(iv) there is evidently a minimal element and we can
assume that the given function $N$ is itself that minimal function.
Under these circumstances it is to be proved that $\sigma = N$ is the real
homomorphism required. The method of proof is the construction
of a function $M$ which fulfills (i)–(iv) and is finally identified with $N$.

We consider the equations (vi) $a\alpha_1 = x^n b\alpha_2 (a, b \in G, \alpha_1 \in H, n \geq 1)$. For each $x$, let $M(x) = \inf \{\phi(h_1)N(a)/\phi(h_2)\}^{1/n}$, the infimum taken
over all solutions of (vi). The verification that $M$ satisfies (ii)–(iv)
follows.

Writing $a = b = h_1 = h_2 = x = 1$ yields $M(1) \leq 1$ and if $a\alpha_1 = b\alpha_2$,
$N(a)\phi(h_1) \geq \phi(h_2)$ by (iv). To prove (iii) for $M$, suppose in addition to
(vi) that $c\alpha_3 = d\alpha_4 h_3$. Then

$$a^m c^n h_1 h_3 = b^m d^n (xy)^m h_2 h_4$$

whence

$$M(xy) \leq \left\{ \frac{\phi(h_1)^m \phi(h_3)^n N(a^m c^n)}{\phi(h_2)^m \phi(h_4)^n} \right\}^{1/mn}$$

which is less than the product of the corresponding ratios for $M(x)$
and $M(y)$.

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For obvious reasons (iv) is attacked indirectly; we show first that $M = \phi$ in $H$. To see that $M \geq \phi$ in $H$ suppose that $ah_1 = h^a h h_2$. Applying (iv) to the function $N$, $N(a)\phi(h_1) \geq \phi(h)\phi(h_2)$ so $M(h) \geq \phi(h)$. In the reverse sense $1 \cdot h = 1 \cdot h \cdot 1$ yields immediately $M(h) \leq \phi(h)$. We observe also that $M(xy) \geq M(x)$, for if $ah_1 = b(xy)^n h_2$ we have $ah_1 = by^n \cdot x^n \cdot h_2$.

The proof that $M$ satisfies (iv) is now easy: $M(a)\phi(h_1) = M(a)M(h_1) \geq M(bh_2) \geq M(h_2)$. Trivially $M \leq N$ so $M = N$.

If $x \in G$ and $m \geq 1$, $M(x^m h) = M(x)^m M(h)$ or equivalently $M(x)^m M(h) \leq M(x^m h)$. To see this suppose $ah_1 = (x^m h)^n h_2$ in (iv); then since $ah_1 = x^{mn} \cdot b \cdot h^n h_2$ $M(x) \leq \{ N(a)\phi(h_1)/\phi(h_2)\phi(h)^n \}^{1/mn}$ which yields the asserted inequality.

Thus $M$ is multiplicative in the semigroup generated by $H \cup \{ x \}$; since $M (= N)$ was minimal, repetition of the same argument shows that $M$ is multiplicative in the semigroup generated by $H \cup \{ x, y \} \cdots$. Since $M(x \cdot 1) \geq M(1) = 1$ is known, the proof of Theorem 1 is complete.

The necessity of (iv), one observes, is obvious.

**Theorem 2.** Suppose $a \in G$, $r > 1$ and $N(a^n x) \geq r^n$ for all $x \in G$, $n \geq 1$. Then there is a real homomorphism $\phi$ of $G$ such that $\phi(a) = r$ and $N \geq \phi \geq 1$.

**Proof.** $H$ is taken here to be the semigroup of 1 and the powers of $a$. Since $N(a^n) \geq r^n$ for each $n$ it is clear $a^m \neq a^n$ when $m > n \geq 0$, for if equality is obtained $H$ would be finite. Thus we can define $\phi(a^n) = r^n$ for $n \geq 0$ and turn to the hypotheses of Theorem 1.

I. $xa^n = ya^m$ and $m = n + p > n \geq 0$. Rearranging, $xa^n = ya^p \cdot a^n$ and then for each integer $s \geq 1$, $x^s a^n = (ya^p)^s \cdot a^n$, inductively: if the last equality holds,

$$x^{s+1} a^n = (ya^p)^s \cdot xa^n = (ya^p)^s+1 a^n.$$  

Then

$$N(x^s a^n) \geq N(y^s a^{p \cdot s+n}) \geq r^{p \cdot s+n}.$$  

Taking the $s$-root and letting $s \to \infty$ gives $N(x) \geq r^n$ or

$$N(x)^r \geq r^{p+n} = r^n.$$  

II. $xa^n = ya^m$ and $n \geq m$. Inasmuch as $r^n \geq r^m$ it is enough to check that $N \geq 1$ in $G$. But

$$N(x)^s N(a) \geq N(x^s a) \geq r > 0$$

so $N(x) \geq 1$. 
Theorem 1 now gives the proof of Theorem 2.

**Theorem 3.** Let $\phi$ be a real homomorphism of $H$ such that $N \geq \phi \geq 0$ and $\phi$ is maximal among all homomorphisms with this property. Then $\phi$ can be extended to a homomorphism $\sigma$ of $G$, $N \geq \sigma \geq 0$.

**Proof.** $\phi$ is maximal also in the subset $S$ of $H$ in which $\phi > 0$. For if $N \geq \chi \geq \phi \geq 0$ in $S$, $\chi$ can be extended to $H$ by setting it equal to zero in $H - S$ so $\chi = \phi$ in $S$. If then $\phi$ can be extended from $S$ to a real homomorphism $\sigma$ of $G$, $N \geq \sigma \geq 0$, necessarily $\sigma \geq \phi$ in $S \cup (H - S)$ so $\sigma = \phi$ in $H$. In brief we can suppose $\phi > 0$ in all of $H$.

The hypothesis that $\phi > 0$ be maximal can be restated: if $N/\phi \geq \rho \geq 1$ for a homomorphism $\rho$ of $H$, then $\rho = 1$. But $N/\phi$ satisfies the axioms (i)–(iii) and Theorem 2 is applicable for $N/\phi = N_1$.

The proof of Theorem 3 is now completed by the criterion of [1] for the existence of the required homomorphism $\sigma$. Namely: $x_0 \in G$, $h_0 \in H$ and $x_0h_0 \in H$, implies that $N(x_0)\phi(h_0) \geq \phi(x_0h_0)$.

The verification follows.

Given $r > 1$ there exist by Theorem 2 an integer $n \geq 1$ and an element $h \in H$ for which $N_1(h_0^n) \leq r^n$ or

$$N(h_0^n) \leq r^n \phi(h_0)^n \phi(h).$$

But $x_0h_0 \in H$ so that

$$0 < \phi(x_0h_0)^n \phi(h) \leq N(x_0)^n N(h_0^n) \leq N(x_0)^n r^n \phi(h_0)^n \phi(h).$$

Finally $0 < \phi(x_0h_0) \leq r N(x_0)\phi(h_0)$ and Theorem 3 is proved.

**Remark.** Hausdorff’s principle may be used to prove the Theorem of [1], as Theorem 1 is proved here. This procedure is used in [2] to obtain a related result.

**References**


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