MAXIMAL SEMICHARACTERS

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In this note the following notations are used.

G denotes a commutative semigroup with identity 1 and H a subsemigroup containing 1.

N is a real-valued function on G such that (i) $N \geqq 0$ (ii) $N(1)=1$ (iii) $N(xy) \leqq N(x)N(y)$ for elements x, y in G.

A *real homomorphism* ϕ is a function on G (or H) satisfying (i), (ii) and $\phi(xy) \equiv \phi(x)\phi(y)$.

Our interest is in real homomorphisms subject to the inequality $N \geqq \phi \geqq 0$, and especially in homomorphisms of this type which are maximal in that $N \geqq \sigma \geqq \phi \geqq 0$ implies $\phi = \sigma$.

THEOREM 1. *Suppose that ϕ is a real homomorphism of H and*
(iv) $ah_1 = bh_2 (a, b \in G, h_i \in H)$ *implies* $N(a)\phi(h_1) \geqq \phi(h_2)$.
Then ϕ can be extended to a real homomorphism σ for which $N \geqq \sigma \geqq 1$.

PROOF. In the family $\{N'\}$ of functions $N' \leqq N$ which fulfill requirements (i)–(iv) there is evidently a minimal element and we can assume that the given function N is itself that minimal function. Under these circumstances it is to be proved that $\sigma \equiv N$ is the real homomorphism required. The method of proof is the construction of a function M which fulfills (i)–(iv) and is finally identified with N.

We consider the equations (vi) $ah_1 = x^n bh_2 (a, b, x \in G, h_i \in H, n \geqq 1)$. For each x, let $M(x) = \inf\{\phi(h_1)N(a)/\phi(h_2)\}^{1/n}$, the infimum taken over all solutions of (vi). The verification that M satisfies (ii)–(iv) follows.

Writing $a = b = h_1 = h_2 = x = 1$ yields $M(1) \leqq 1$ and if $ah_1 = bh_2$, $N(a)\phi(h_1) \geqq \phi(h_2)$ by (iv). To prove (iii) for M, suppose in addition to (vi) that $ch_3 = dy^m h_4$. Then

$$a^m c^n \cdot h_1^m h_3^n = b^m d^n \cdot (xy)^{mn} h_2^m h_4^n$$

whence

$$M(xy) \leqq \left\{ \frac{\phi(h_1)^m \phi(h_3)^n N(a^m c^n)}{\phi(h_2)^m \phi(h_4)^n} \right\}^{1/mn}$$

which is less than the product of the corresponding ratios for $M(x)$ and $M(y)$.

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For obvious reasons (iv) is attacked indirectly; we show first that $M \equiv \phi$ in H. To see that $M \geqq \phi$ in H suppose that $ah_1 = h^n b h_2$. Applying (iv) to the function N, $N(a)\phi(h_1) \geqq \phi(h)^n \phi(h_2)$ so $M(h) \geqq \phi(h)$. In the reverse sense $1 \cdot h = 1 \cdot h \cdot 1$ yields immediately $M(h) \leqq \phi(h)$. We observe also that $M(xy) \geqq M(x)$, for if $ah_1 = b(xy)^n h_2$ we have $ah_1 = by^n \cdot x^n \cdot h_2$.

The proof that M satisfies (iv) is now easy: $M(a)\phi(h_1) = M(a)M(h_1) \geqq M(bh_2) \geqq M(h_2)$. Trivially $M \leqq N$ so $M = N$.

If $x \in G$ and $m \geqq 1$, $M(x^m h) = M(x)^m M(h)$ or equivalently $M(x)^m M(h) \leqq M(x^m h)$. To see this suppose $ah_1 = (x^m h)^n b h_2$ in (iv); then since $ah_1 = x^{mn} \cdot b \cdot h^n h_2$ $M(x) \leqq \{N(a)\phi(h_1)/\phi(h_2)\phi(h)^n\}^{1/mn}$ which yields the asserted inequality.

Thus M is multiplicative in the semigroup generated by $H \cup \{x\}$; since M $(=N)$ was minimal, repetition of the same argument shows that M is multiplicative in the semigroup generated by $H \cup \{x, y\}$ $\cdots$. Since $M(x \cdot 1) \geqq M(1) = 1$ is known, the proof of Theorem 1 is complete.

The necessity of (iv), one observes, is obvious.

THEOREM 2. *Suppose* $a \in G$, $r > 1$ *and* $N(a^n x) \geqq r^n$ *for all* $x \in G$, $n \geqq 1$. *Then there is a real homomorphism* ϕ *of* G *such that* $\phi(a) = r$ *and* $N \geqq \phi \geqq 1$.

PROOF. H is taken here to be the semigroup of 1 and the powers of a. Since $N(a^n) \geqq r^n$ for each n it is clear $a^m \neq a^n$ when $m > n \geqq 0$, for if equality is obtained H would be finite. Thus we can define $\phi(a^n) = r^n$ for $n \geqq 0$ and turn to the hypotheses of Theorem 1.

I. $xa^n = ya^m$ and $m = n + p > n \geqq 0$. Rearranging, $xa^n = ya^p \cdot a^n$ and then for each integer $s \geqq 1$, $x^s a^n = (ya^p)^s \cdot a^n$, inductively: if the last equality holds,

$$x^{s+1}a^n = (ya^p)^s \cdot xa^n = (ya^p)^{s+1}a^n.$$

Then

$$N(x)^s N(a)^n \geqq N(y^s a^{ps+n}) \geqq r^{ps+n}.$$

Taking the s-root and letting $s \to \infty$ gives $N(x) \geqq r^p$ or

$$N(x)r^n \geqq r^{p+n} = r^m.$$

II. $xa^n = ya^m$ and $n \geqq m$. Inasmuch as $r^n \geqq r^m$ it is enough to check that $N \geqq 1$ in G. But

$$N(x)^s N(a) \geqq N(x^s a) \geqq r > 0$$

so $N(x) \geqq 1$.

Theorem 1 now gives the proof of Theorem 2.

THEOREM 3. *Let ϕ be a real homomorphism of H such that $N \geqq \phi \geqq 0$ and ϕ is maximal among all homomorphisms with this property. Then ϕ can be extended to a homomorphism σ of G, $N \geqq \sigma \geqq 0$.*

PROOF. ϕ is maximal also in the subset S of H in which $\phi > 0$. For if $N \geqq \chi \geqq \phi \geqq 0$ in S, χ can be extended to H by setting it equal to zero in $H-S$ so $\chi = \phi$ in S. If then ϕ can be extended from S to a real homomorphism σ of G, $N \geqq \sigma \geqq 0$, necessarily $\sigma \geqq \phi$ in $S \cup (H-S)$ so $\sigma = \phi$ in H. In brief we can suppose $\phi > 0$ in all of H.

The hypothesis that $\phi > 0$ be maximal can be restated: if $N/\phi \geqq \rho \geqq 1$ for a homomorphism ρ of H, then $\rho = 1$. But N/ϕ satisfies the axioms (i)–(iii) and Theorem 2 is applicable for $N/\phi = N_1$.

The proof of Theorem 3 is now completed by the criterion of [1] for the existence of the required homomorphism σ. Namely: $x_0 \in G$, $h_0 \in H$ and $x_0 h_0 \in H$, implies that $N(x_0)\phi(h_0) \geqq \phi(x_0 h_0)$.

The verification follows.

Given $r > 1$ there exist by Theorem 2 an integer $n \geqq 1$ and an element $h \in H$ for which $N_1(h_0^n h) \leqq r^n$ or

$$N(h_0^n h) \leqq r^n \phi(h_0)^n \phi(h).$$

But $x_0 h_0 \in H$ so that

$$0 < \phi(x_0 h_0)^n \phi(h) \leqq N(x_0)^n N(h_0^n h) \leqq N(x_0)^n r^n \phi(h_0)^n \phi(h).$$

Finally $0 < \phi(x_0 h_0) \leqq r N(x_0) \phi(h_0)$ and Theorem 3 is proved.

REMARK. Hausdorff's principle may be used to prove the Theorem of [1], as Theorem 1 is proved here. This procedure is used in [2] to obtain a related result.

REFERENCES

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