EXTREME POINTS OF BOUNDED ANALYTIC FUNCTIONS ON INFINITELY CONNECTED REGIONS[1]

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1. Introduction. For R a region in the complex plane, let $H^\infty(R)$ be the Banach space of bounded analytic functions on R with the supremum norm:

$$\|f\| = \sup_R |f|.$$

We will describe the extreme points of the unit ball of $H^\infty(R)$ for a certain class of regions.

When R is the unit disk, D, the result is well known. A proof is in [**3**, pp. 138–139].

THEOREM 1. *Suppose $f \in H^\infty(D)$ and $\|f\| \leqq 1$. Then f is an extreme point of the unit ball of $H^\infty(D)$ if, and only if,*

$$\int_0^{2\pi} \log (1 - |f(e^{i\theta})|) d\theta = -\infty.$$

We will consider a region R which can be represented as follows: $R = D - (\bigcup_1^\infty K_j)$ where:

(1) $\{K_j | j = 1, 2, \cdots\}$ is a collection of pairwise disjoint connected compact subsets of D;

(2) there are points $w_j \in K_j$, $j = 1, 2, \cdots$, such that

$$\sum 1 - |w_j| < \infty.$$

Let $T: D \to R$ be an analytic universal covering map of D onto R and let Γ be the group of fractional linear transformations, γ, taking D onto D such that $T \circ \gamma = T$. (See [**6**, Chapter 9].) Let H^∞/Γ be the closed subspace of $H^\infty(D)$ consisting of all $f \in H^\infty(D)$ such that $f \circ \gamma = f$ for all $\gamma \in \Gamma$. Then $F \to F \circ T$ is an isometric isomorphism of $H^\infty(R)$ onto H^∞/Γ.

THEOREM 2. *Suppose $f \in H^\infty/\Gamma$ and $\|f\| \leqq 1$. Then f is an extreme point of the unit ball of H^∞/Γ if, and only if,*

$$\int_0^{2\pi} \log (1 - |f(e^{i\theta})|) d\theta = -\infty.$$

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In §3 we will show that Theorem 2 holds for a more general class of regions.

It should be noted that Gamelin in [2] characterized the extreme points of the unit ball of $H^\infty(R)$ when R is a finite bordered Riemann surface.

2. Proof of Theorem 2.

LEMMA 1. (a) *For* $0<r<1$, $\{|z|\leqq r\}$ *intersects only a finite number of the* K_j*'s.* (b) *There are simple closed curves* α_j *in* D, $j=1, 2, \cdots$, *such that* K_j *is interior to* α_j *and* $\bigcup_{k\neq j}K_k$ *is exterior to* α_j.

PROOF. Let $K_0=\{|z|=1\}$ and $X=\bigcup_0^\infty K_j$. We show first that each K_j, $j=0, 1, \cdots$, is a component of X. Suppose $x_0\in K_k$ and K is the component of X containing x_0. Then $K_k\subset K$ and $K=\bigcup_0^\infty(K\cap K_j)$. Since K is compact and connected, and each intersection, $K\cap K_j$, is compact, $K\cap K_j$ is empty for $j\neq k$ [4, p. 170]. So $K=K_k$.

Let $X_r=X\cap\{|z|\leqq r\}$. Since no component of X meets both K_0 and X_r there are disjoint closed sets H_1 and H_2 such that $X=H_1\cup H_2$, $X_r\subset H_1$ and $K_0\subset H_2$ [5, p. 82]. Then $H_1\subset\{|z|\leqq r_1\}$ for some $r\leqq r_1<1$, so $K_j\subset H_2$ for large j since $|w_j|\to 1$ where $w_j\in K_j$. This proves (a).

Part (a) implies that K_j and $\bigcup_{k\neq j}K_k$ are separated in D which implies (b).

PROOF OF THEOREM 2. As is well known, f is an extreme point if, and only if, the following condition holds: if $g\in H^\infty/\Gamma$, $\|f+g\|\leqq 1$ and $\|f-g\|\leqq 1$ then $g=0$ [3, p. 138].

Since H^∞/Γ is a subspace of $H^\infty(D)$ the sufficiency of the integral condition is implied by Theorem 1.

To show that the integral condition is necessary suppose $\log(1-|f(e^{i\theta})|)$ is integrable. Let

$$h(z) = \exp\left[\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{i\theta}+z}{e^{i\theta}-z}\log(1-|f(e^{i\theta})|)d\theta\right].$$

Then h is an outer function in $H^\infty(D)$ [3, p. 62]. Also $|h(e^{i\theta})| = 1-|f(e^{i\theta})|$ a.e. and so $|h(\gamma(e^{i\theta}))|=|h(e^{i\theta})|$ a.e. for all $\gamma\in\Gamma$ and it follows that

$$(*)\qquad |h(\gamma(z))| = |h(z)|,\qquad z\in D,\ \gamma\in\Gamma.$$

Let $H=h\circ T^{-1}$. Then H is a bounded multiple-valued analytic function on R which never vanishes. Property (*) implies that $|H|$ is single-valued [8, Lemma 3.5].

For each $j=1, 2, \cdots$, let α_j be a simple closed positively oriented curve in R which separates K_j and $\bigcup_{k\neq j}K_k$. Let p_j be the period of log H over α_j. Let r_j be the constant such that $0 \leqq r_j < 1$ and $-p_j = 2\pi r_j i$ modulo $2\pi i$, $j=1, 2, \cdots$.

Now since $\sum 1 - |w_j| < \infty$ it follows that

$$-\infty < \sum \log \left| \frac{z - w_j}{1 - \bar{w}_j z} \right|$$

for $z \in D - \{w_j | j=1, 2, \cdots\}$. Then

$$u(z) = \sum \log \left| \frac{z - w_j}{1 - \bar{w}_j z} \right|^{r_j}$$

is harmonic and nonpositive on R. Let v be the harmonic conjugate of u on R and $P = \exp(u + iv)$. Then P is a multiple-valued analytic function on R such that $|P|$ is single-valued and $|P| \leqq 1$ on R. The period of $u + iv$ over α_j is $2\pi r_j i$ so HP is single-valued on R. Hence for $p = P \circ T$ we have $|p| \leqq 1$ and $g = hp \in H^\infty/\Gamma$. Thus on $|z| = 1$, $|g| = |hp| \leqq |h| = 1 - |f|$. That is, $|f| + |g| \leqq 1$ on $|z| = 1$. This implies that $\|f + g\| \leqq 1$ and $\|f - g\| \leqq 1$. Since g is not the zero function, f is not an extreme point. The proof is complete.

3. **More general regions.** Theorem 2 holds for a more general class of regions; namely when R is a region on a finite bordered Riemann surface, R_0, such that $R = R_0 - (\bigcup_1^\infty K_j)$ where:

(1′) $\{K_j | j=1, 2, \cdots\}$ is a collection of pairwise disjoint connected compact subsets of R_0;

(2′) there are points $w_j \in K_j$, $j=1, 2, \cdots$, such that $\sum g(z, w_j) < \infty$ where g is the Green's function for R_0.

Note that when $R_0 = D$ conditions (2) and (2′) are equivalent.

The proof of Theorem 2 in this more general setting is essentially the same as before with some added complications. We will outline the procedure. Let $C_1, \cdots, C_m$ be the components of the border of R_0 and ϕ_j, $j=1, \cdots, m$, conformal maps of an annulus $A = \{r < |z| < 1\}$ into R_0 such that $\phi_j(\{|z| = 1\}) = C_j$, and the images, $\phi_1(A), \cdots, \phi_m(A)$, are disjoint. Then arguments similar to those in Lemma 1 show that only a finite number of the K_j's meet $R_0 - (\bigcup_1^m \phi_j(A))$. Suppose for all $j \geqq N+1$, K_j does not meet this set. Let $K = \bigcup_1^N K_j$. Then one can show that the first singular homology group of $R_0 - K$ with integer coefficients is finitely generated using the homology sequence of $(R_0, R_0 - K)$, and the Alexander duality theorem. It then follows that $R_0 - K$ is conformally equivalent to a finite bordered Riemann surface with a finite number of points deleted [**7**, Theorem 8.1]. So we can assume to begin with that each K_j is included in a conformal image, N_j, of the unit disk such that $K_k \cap N_j$ is empty for $k \neq j$. Let α_j be a positively oriented simple closed curve in N_j containing K_j in its interior region. Proceeding as in the proof of Theorem 2 we must show that for H a bounded multiple-valued analytic function on R such that H never vanishes and $|H|$ is single-valued there is a multiple-valued analytic function, P, on R with $|P|$ single-valued and $|P| \leqq 1$ such that HP is single-valued.

Let $\beta_1, \cdots, \beta_s$ be a homology basis for R_0. We can assume no β_k intersects any K_j. Then $\beta_1, \cdots, \beta_s, \alpha_1, \alpha_2, \cdots$ is a homology basis for R. Let p_j be the period of log H over α_j, $j=1, 2, \cdots$ and $0 \leqq r_j < 1$ such that $-p_j = 2\pi r_j i$ modulo $2\pi i$. Then $-\sum r_j g(z, w_j)$ is harmonic and nonpositive on $R_0 - \{w_j | j=1, 2, \cdots\}$. Let q_j be the period of the harmonic conjugate of $(\log |H|) - \sum r_j g(z, w_j)$ over β_j, $j=1, 2, \cdots, s$. There is a negative harmonic function W on R_0 whose harmonic conjugate has periods $-q_j$ over β_j, $j=1, 2, \cdots, s$ [1, Theorem 4]. Let $u = W - \sum r_j g(z, w_j)$ and v be the harmonic conjugate of u on R. Then $P = \exp(u+iv)$ is the desired function.

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