COMMUTATIVE, NOETHERIAN RINGS OVER WHICH EVERY MODULE HAS A MAXIMAL SUBMODULE
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In [1, p. 470] Professor Hyman Bass mentions the following conjecture: a ring $R$ is left perfect if, and only if, every nonzero left $R$-module has a maximal submodule and $R$ has no infinite set of orthogonal idempotents. If a ring $R$ is right or left noetherian, then $R$ has no infinite set of orthogonal idempotents. We shall show that for commutative, noetherian rings Bass' conjecture is true.

**Lemma.** If $R$ is a commutative ring over which every nonzero module has a maximal submodule, then every proper prime ideal of $R$ is maximal.

**Proof.** Let $P$ be a proper prime ideal of $R$ so $S = R/P$ is an integral domain over which every nonzero module has a maximal submodule. Let $sQ$ be a nonzero, injective $S$-module. Then $sQ$ has a simple epimorphic image, say $S/M$ where $M$ is a maximal ideal of $S$. If $m \in M$ and $m \neq 0$, then, being the quotient of an injective module, $S/M$ is divisible, and there is an $s \in S$ with $m(s + M) = ms + M = 1 + M$. Hence $1 \in M$, a contradiction. Thus $M = 0$, and $S$ is a field.

If $RM$ is an $R$-module, then $RM$ is $R$-projective if for each epimorphism $\sigma : R \rightarrow C$ and each homomorphism $\pi : R \rightarrow C$, there is a homomorphism $\tau : RM \rightarrow R$ such that $\tau \sigma = \pi$. We call a ring $R$ a test module for projectivity if every $R$-projective module is projective.

**Theorem 1.** Let $R$ be a commutative, noetherian ring. Then the following are equivalent.

(i) $R$ is a test module for projectivity,
(ii) every nonzero $R$-module has a maximal submodule, and
(iii) $R$ is artinian.

**Proof.** (iii) ⇒ (i). As Professor Barbara L. Osofsky shows in [3], this implication follows from Sandomierski [4, Theorems 4.1 and 4.4].
(i) ⇒ (ii): Every nonzero projective module has a nonzero homo-
morphic image in a cyclic. If a module $M$ has no nonzero homomorphic image in a cyclic, $M$ is trivially $R$-projective so by (i) $M$ is projective and $M = 0$. Thus, if $M$ is a nonzero module, there exists a nonzero homomorphism $\sigma$ of $M$ to a cyclic. Then $\text{Im} \sigma$ is finitely generated, so $\text{Im} \sigma$ has a maximal submodule, and so must $M$.

(ii) $\Rightarrow$ (iii). It suffices to show that $R$ is perfect by a remark in Bass [1, p. 475]. Since $R$ has no infinite set of orthogonal idempotents, $R$ is perfect if every nonzero $R$-module has a simple submodule [1]. Let $M$ be a nonzero $R$-module with $m \in M$, $m \neq 0$. Select a maximal ideal from $\{(0:rm) : r \in R, rm \neq 0\}$, say $(0:sm)$. Suppose that $ab \in (0:sm)$, but $a \notin (0:sm)$. Then $asm \neq 0$ and $(0:sm) \subset (0:asm)$ imply $(0:sm) = (0:asm)$. This shows that $b \in (0:sm)$, and $(0:sm)$ is a proper prime ideal of $R$. By the lemma $(0:sm)$ is maximal so $Rsm$ is simple.

**Theorem 2.** A commutative ring $R$ is perfect if, and only if, every nonzero $R$-module has a maximal submodule and $R/J$ (where $J$ is the Jacobson radical of $R$) satisfies the ascending chain condition on the annihilators of principal ideals.

**Proof.** Clearly if $R$ is perfect the second part of the theorem holds; see [1]. Conversely, if the second part of the theorem holds, then by a remark in [1, p. 470] to show $R$ is perfect it suffices to show that $R/J$ is semi-simple artin. The condition on annihilator ideals implies that $R/J$ has no infinite set of orthogonal idempotents. Then using the obvious modification of the technique in (ii) $\Rightarrow$ (iii) above, we obtain that $R/J$ is an essential extension of its socle. Thus $R/J$ is semi-simple artin.

**References**