COMMUTATIVE, NOETHERIAN RINGS OVER WHICH EVERY MODULE HAS A MAXIMAL SUBMODULE

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In [1, p. 470] Professor Hyman Bass mentions the following conjecture: a ring R is left perfect if, and only if, every nonzero left R-module has a maximal submodule and R has no infinite set of orthogonal idempotents. If a ring R is right or left noetherian, then R has no infinite set of orthogonal idempotents. We shall show that for commutative, noetherian rings Bass' conjecture is true.

LEMMA. *If R is a commutative ring over which every nonzero module has a maximal submodule, then every proper prime ideal of R is maximal.*

PROOF. Let P be a proper prime ideal of R so $S=R/P$ is an integral domain over which every nonzero module has a maximal submodule. Let ${}_SQ$ be a nonzero, injective S-module. Then ${}_SQ$ has a simple epimorphic image, say S/M where M is a maximal ideal of S. If $m\in M$ and $m\neq 0$, then, being the quotient of an injective module, S/M is divisible, and there is an $s\in S$ with $m(s+M)=ms+M=1+M$. Hence $1\in M$, a contradiction. Thus $M=0$, and S is a field.

If ${}_RM$ is an R-module, then ${}_RM$ is R-projective if for each epimorphism $\sigma: {}_RR\rightarrow {}_RC$ and each homomorphism $\pi: {}_RM\rightarrow {}_RC$, there is a homomorphism $\tau: {}_RM\rightarrow {}_RR$ such that $\tau\sigma=\pi$. We call a ring R a test module for projectivity if every R-projective module is projective.

THEOREM 1. *Let R be a commutative, noetherian ring. Then the following are equivalent.*

(i) *R is a test module for projectivity,*

(ii) *every nonzero R-module has a maximal submodule, and*

(iii) *R is artinian.*

PROOF. (iii)⇒(i). As Professor Barbara L. Osofsky shows in [3], this implication follows from Sandomierski [4, Theorems 4.1 and 4.4].

(i)⇒(ii): Every nonzero projective module has a nonzero homo-

Received by the editors June 6, 1966.

[1] This note forms a part of the author's Henry Rutgers thesis written at Rutgers, the State University, under the direction of Professor Barbara L. Osofsky on a project sponsored by the National Science Foundation through Undergraduate Research Participation Grants GE-1886, GE-4206 and GE-6287.

morphic image in a cyclic. If a module M has no nonzero homomorphic image in a cyclic, M is trivially R-projective so by (i) M is projective and $M=0$. Thus, if M is a nonzero module, there exists a nonzero homomorphism σ of M to a cyclic. Then Im σ is finitely generated, so Im σ has a maximal submodule, and so must M.

(ii)$\Rightarrow$(iii). It suffices to show that R is perfect by a remark in Bass [**1**, p. 475]. Since R has no infinite set of orthogonal idempotents, R is perfect if every nonzero R-module has a simple submodule [**1**]. Let M be a nonzero R-module with $m \in M$, $m \neq 0$. Select a maximal ideal from $\{(0:rm): r \in R,\ rm \neq 0\}$, say $(0:sm)$. Suppose that $ab \in (0:sm)$, but $a \notin (0:sm)$. Then $asm \neq 0$ and $(0:sm) \subset (0:asm)$ imply $(0:sm) = (0:asm)$. This shows that $b \in (0:sm)$, and $(0:sm)$ is a proper prime ideal of R. By the lemma $(0:sm)$ is maximal so Rsm is simple.

THEOREM 2. *A commutative ring R is perfect if, and only if, every nonzero R-module has a maximal submodule and R/J (where J is the Jacobson radical of R) satisfies the ascending chain condition on the annihilators of principal ideals.*

PROOF. Clearly if R is perfect the second part of the theorem holds; see [**1**]. Conversely, if the second part of the theorem holds, then by a remark in [**1**, p. 470] to show R is perfect it suffices to show that R/J is semi-simple artin. The condition on annihilator ideals implies that R/J has no infinite set of orthogonal idempotents. Then using the obvious modification of the technique in (ii)$\Rightarrow$(iii) above, we obtain that R/J is an essential extension of its socle. Thus R/J is semi-simple artin.

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