ORLICZ SPACES ISOMORPHIC TO STRICTLY CONVEX SPACES[1]

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In this paper it is proved that if (X, S, μ) is a measure space then the Orlicz space $L_\Phi(X, S, \mu)$ is isomorphic to a strictly convex Banach space if the Young function Φ satisfies the growth condition G_1 or G_2 defined below according as $\mu(X)$ is infinite or $\mu(X)$ is finite. As corollaries of this theorem we obtain (1) a simple proof of the known theorem, Day [2], that every L-space is isomorphic to a strictly convex space, and (2) every reflexive Orlicz space is isomorphic to a strictly convex space if (X, S, μ) is σ-finite.

DEFINITION 1. If Φ is a Young function and (X, S, μ) is a measure space then $L_\Phi(X, S, \mu)$ is the linear space of all complex valued μ-measurable functions f such that $\Phi(|kf|)$ is μ-summable for some nonzero real number k after the usual identification that $f=g$ if and only if $f=g$ a.e.

It is known (Weiss [5]) that $L_\Phi(X, S, \mu)$ is a Banach space under the norm $\|\ \|_\Phi$ defined by

$$\|f\|_\Phi = \inf \left\{1/a \mid a \geqq 0, \int_X \Phi(|af|)d\mu \leqq 1\right\}.$$

In what follows we assume Φ is a nonzero Young function and L_Φ is the Banach space $(L_\Phi(X, S, \mu), \|\ \|_\Phi)$.

DEFINITION 2. The Young function Φ is said to satisfy the condition G_1 if there exists a constant C such that $\Phi(2u) \leqq C\Phi(u)$ for all $u \geqq 0$, and Φ is said to satisfy G_2 if the above inequality is true for large u.

REMARK 1. If $\mu(X)$ is infinite and Φ satisfies the condition G_1 then the integral $M(f) = \int_X \Phi(|f|)d\mu$ is finite for all f in L_Φ. If $\mu(X)$ is finite then the same is true if Φ satisfies the condition G_2. Further it is verified that if $M(f)$ is finite for all f in L_Φ then $\|f\|_\Phi = 1$ if and only if $M(f) = 1$.

We establish two lemmas before proceeding to the main theorem.

LEMMA 1. *If Φ is a Young function then there exists a Young function Φ_1 such that* (i) $\Phi(u) \leqq \Phi_1(u) \leqq 2\Phi(u)$ *and* (ii) $\Phi_1(u)$ *is strictly convex in* $]v, \infty[$ *where* $v = \sup \Phi^{-1}(0)$.

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PROOF. It suffices to construct a Young function Φ_1 satisfying the inequality (i) in the lemma with its left derivative strictly increasing in the interval $]v, \infty[$.

First we note that since Φ is a Young function, the left derivative of Φ, say, ϕ exists at all points in $]0, \infty[$ and further Φ admits the integral representation $\Phi(u)=\int_{[0,u]}\phi d\lambda$ where λ is the Lebesgue measure.

Let $\phi_0(t)$ be the real valued function defined on $[0, \infty[$ by setting

$$\phi_0(t) = \frac{t}{2}\phi(t)$$

if $0\leqq t\leqq 1$, and

$$\phi_0(t) = \left(\sum_{i=1}^{n-1}\frac{1}{2^2}\phi(i)\right) + \frac{t-(n-1)}{2^n}\phi(t)$$

if $n-1\leqq t\leqq n$ where the integer $n\geqq 2$. Since ϕ is nondecreasing, ϕ_0 is nondecreasing and $\phi_0(t)\leqq\phi(t)$ for $t\geqq 0$. Further (a) $\phi_0(t)=0$ if and only if $\phi(t)=0$ and (b) $\phi_0(t_1)\neq 0\neq\phi_0(t_2)$ and $t_1>t_2$ imply $\phi_0(t_1)>\phi_0(t_2)$. Now consider the Young function defined by

$$\Phi_0(u) = \int_{[0,u]}\phi_0(t)d\lambda.$$

It follows from (a) above that $\Phi_0(u)=0$ if and only if $\Phi(u)=0$. Hence from (b) it follows that ϕ_0 is strictly increasing in the interval $]v, \infty[$. Thus Φ_0 is strictly convex in $]v, \infty[$. Now the Young function $\Phi_1=\Phi+\Phi_0$ has the desired properties.

LEMMA 2. *If the Young function Φ satisfies the condition G_2 and $\mu(X)$ is finite then there exists a Young function Φ' such that $\Phi'(u)>0$ for all $u>0$ and L_Φ is isomorphic to $L_{\Phi'}$.*

PROOF. Since Φ satisfies the condition G_2 there exist two nonnegative real numbers C and u_0 such that for all $u\geqq u_0$, $\Phi(2u)\leqq C\Phi(u)$. Further since Φ is nonzero, $C\geqq 2$. Let Φ' be the function on $[0, \infty[$ defined by setting

$$\Phi'(u) = \sum_{n\geqq 0}\frac{1}{C^{2n}}\Phi(2^n u).$$

We prove first that $\Phi'(u)<\infty$ for all $u\geqq 0$. Let $u_1\geqq 0$. Let N be the smallest integer $\geqq 0$ such that $2^N u_1\geqq u_0$. Then it is verified that

$$\Phi'(u_1) = \sum_{n<N} \frac{1}{C^{2n}} \Phi(2^n u_1) + \sum_{n \geqq N} \frac{1}{C^{2n}} \Phi(2^n u_1)$$

$$\leqq \sum_{n<N} \frac{1}{C^{2n}} \Phi(2^n u_1) + \sum \frac{c^{n-N}}{C^{2n}} \Phi(2^N u_1)$$

$$\leqq \sum_{n<N} \frac{1}{C^{2n}} \Phi(2^n u_1) + \frac{C}{C-1} \Phi(2^n u_1).$$

Hence $\Phi'(u_1)$ is finite. Thus $\Phi'(u)$ is a convex real valued function defined on $[0, \infty[$ with $\Phi'(0)=0$ and Φ' is a Young function. Next let $f \in L_\Phi$. From Remark 1 it follows that $\Phi(|f|)$ is μ-summable and if $E_1 = \{x \mid |f(x)| \geqq u_0\}$ and $E_2 = X \sim E_1$, then

$$\int_X \Phi'(|f|)d\mu \leqq \Phi'(u_0)\mu(E_2) + \frac{C}{C-1} \int_{E_1} \Phi(|f|)d\mu$$

since $\Phi(2u) \leqq C\Phi(u)$ for $u \geqq u_0$. Thus $\Phi'(|f|)$ is also μ-summable and $f \in L_{\Phi'}$. Thus L_Φ, $L_{\Phi'}$ are Banach spaces consisting of the same functions and $\|f\|_\Phi \leqq \|f\|_{\Phi'}$. Hence L_Φ is isomorphic to $L_{\Phi'}$. Since $\Phi(u) > 0$ for large u, $\Phi'(u) > 0$ for all $u > 0$.

REMARK 2. Since Φ satisfies the condition G_2 the function Φ_1 also satisfies the condition G_2.

THEOREM 1. *If Φ satisfies the condition G_1 then L_Φ is isomorphic to a strictly convex space. If $\mu(X)$ is finite and Φ satisfies G_2 then L_Φ is isomorphic to a strictly convex space.*

PROOF. Let Φ satisfy the condition G_2. Hence $\Phi(u) > 0$ for all $u > 0$. Hence by Lemma 1 there exists a strictly convex Young function Φ_1 such that $\Phi(u) \leqq \Phi_1(u) \leqq 2\Phi(u)$. Thus L_Φ and L_{Φ_1} are isomorphic under the identity mapping. Further since Φ satisfies the condition G_1, Φ_1 also satisfies the condition G_1. Thus $\int_X \Phi_1(|f|)d\mu < \infty$ for all $f \in L_{\Phi_1}$. Hence $\|f\|_{\Phi_1} = 1$ if and only if $\int_X \Phi(|f|)d\mu = 1$. Thus if $\|f\|_{\Phi_1} = \|g\|_{\Phi_1} = \|(f+g)/2\|_{\Phi_1} = 1$ then

$$\int_X \left[\Phi_1\left(\left| \frac{f+g}{2} \right| \right) - \frac{1}{2} \Phi_1(|f|) - \frac{1}{2} \Phi_1(|g|) \right] d\mu = 0.$$

Since Φ_1 is strictly convex and $\Phi_1(u) > 0$ for $u > 0$ the last equation implies $f = g$ a.e.

Next let $0 \leqq \mu(X) < \infty$ and Φ satisfy the condition G_2. Then by Lemma 2 there exists an Young function Φ_1 such that L_Φ is isomorphic with L_{Φ_1} where $\Phi_1(u)>0$ for all $u>0$. Now the first part of Theorem 1 applies to L_{Φ_1} completing the proof of the second part.

Since the above theorem is applicable when $\Phi(u)=u$ the following corollary is obtained.

COROLLARY 1 [DAY]. *Every L-space is isomorphic to a strictly convex space.*

It is not known, Day [3], whether every reflexive Banach space is isomorphic to a strictly convex space. The following corollary settles the problem for Orlicz spaces when the underlying measure space is σ-finite.

COROLLARY 2. *If (X, S, μ) is σ-finite and L_Φ is a reflexive space then L_Φ is isomorphic to a strictly convex space.*

PROOF. It follows from Theorems 5 and 6 on pages 60 and 61 of Luxemburg [4], that L_Φ is reflexive implies one of the following three statements if (X, S, μ) is σ-finite.

(A) Φ satisfies the condition G_1.

(B) $0<\mu(X)<\infty$ and Φ satisfies the condition G_2.

(C) L_Φ is a separable Banach space.

If we are in the cases (A) and (B) then Theorem 1 is applicable and L_Φ is isomorphic to a strictly convex space. If we are in the case (C) then by a well-known theorem of Clarkson [1], L_Φ is isomorphic to a strictly convex space.

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